The Michael-Simon-Sobolev inequality on manifolds for positive symmetric tensor fields

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Abstract

We prove the Michael-Simon-Sobolev inequality for smooth symmetric uniformly positive definite $(0,2)$-tensor fields on compact submanifolds with or without boundary in Riemannian manifolds with nonnegative sectional curvature by the Alexandrov-Bakelman-Pucci (ABP) method. It should be a generalization of S. Brendle in [**?**].

Keywords: the Michael-Simon-Sobolev inequality; compact submanifold; nonnegative sectional curvature.

1 Introduction

Geometric inequalities are indispensable tools in geometric analysis, serving as vital bridges to understanding, comparing, and quantifying geometric quantities. They empower mathematicians to prove theorems, derive properties, and solve complex problems. Two notable examples are the LiYau estimate and the Sobolev inequality. The LiYau estimate is a powerful tool for estimating the decay rate of solutions to the heat equation. Recent advancements were made by Q. S. Zhang and X. L. Li, who introduced matrix LiYauHamilton estimates for positive solutions to the heat equation and the backward conjugate heat equation under Ricci flow (see [**?**]), their work builds upon earlier contributions (see [**?**, **?**, **?**, **?**, **?**, **?**]). In parallel, numerous experts have undertaken further research into the Sobolev inequality (see [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**]). In 2018, D. Serre (see [**?**]) established a new sharp Sobolev type inequality on a bounded convex domain Ω for a symmetric positive semi-definite (0,2)-tensor field in $\mathbb{R}^n$. In 2019, S. Brendle (see [**?**]) proved a new Michael-Simon-Sobolev inequality for functions on compact submanifolds of arbitrary dimension and codimension in Euclidean space using the Alexandrov-Bakelman-Pucci (ABP) method, the inequality is sharp when the codimension is at most 2, resolving a long-standing conjecture. Building upon this work, S. Brendle (see [**?**] or [**?**]) extended his findings in [**?**] to the scenario where the ambient space is a Riemannian manifold with nonnegative curvature. Inspired by the contributions of D. Serre (see [**?**]) and S. Brendle (see [**?**]), D. Pham (see [**?**]) established a Sobolev inequality involving a positive symmetric matrix-valued function A on a smooth bounded convex domain in $\mathbb{R}^n$, albeit without the convexity condition of Ω.

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In this paper, inspired by [**?**] and [**?**], we present a new Michael-Simon type Sobolev inequality which replaces the positive symmetric matrix-valued function in [**?**] with a smooth symmetric uniformly positive definite $(0,2)$-tensor field on a compact submanifold in Riemannian manifolds with nonnegative sectional curvature using the ABP method. Our result represents a generalization of both D. Serres inequality and a extension of S. Brendles inequality.

Let $(M,\bar{g})$ be a complete noncompact Riemannian manifold of dimension k with nonnegative Ricci curvature. Denote by $\left|B^k\right|$ the volume of the unit ball in $\mathbb{R}^k$. The asymptotic volume ratio of M is defined as

$$\theta := \lim_{r\to\infty} \frac{\{p \in M : d(p,q) \leq r\}}{|B^k| r^k}$$

where q is some fixed point on the manifold. According to the Bishop-Gromov volume comparison theorem, the limit exists and that $\theta \leq 1$.

We get the following principal theorem:

Theorem 1.1. Let $(M,\bar{g})$ be a complete noncompact manifold of dimension $n+m$ with nonnegative sectional curvature, where $m \geq 2$. Let Σ be a compact n-dimensional submanifold of M^{n+m} with smooth boundary $\partial\Sigma$ (possibly $\partial\Sigma = \varnothing$). If A is a smooth symmetric uniformly positive definite $(0,2)-$tensor field on Σ, then

$$\int_\Sigma \sqrt{\left|\mathrm{div}_\Sigma A\right|^2 + \left|\langle A, II\rangle\right|^2} + \int_{\partial\Sigma} |A(\nu)| \geq n \left[\frac{(n+m)\left|B^{n+m}\right|}{m\left|B^m\right|}\right]^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\int_\Sigma (\det A)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}, \quad (1.1)$$

where ν is the unit outer normal vector field on $\partial\Sigma$ with respect to Σ, II is the second fundamental form of Σ, and θ is the asymptotic volume ratio of M.

When $m=2$, because of $(n+2)\left|B^{n+2}\right| = 2\left|B^2\right|\left|B^n\right|$, we establish a sharp Sobolev inequality for submanifolds of codimensional 2 as follows:

Corollary 1.2. Let $(M,\bar{g})$ be a complete noncompact manifold of dimension $n+2$ with nonnegative sectional curvature. Let Σ be a compact n-dimensional submanifold of M^{n+2} with smooth boundary $\partial\Sigma$ (possibly $\partial\Sigma = \varnothing$). If A is a smooth symmetric uniformly positive definite $(0,2)-$tensor field on Σ, then

$$\int_\Sigma \sqrt{\left|\mathrm{div}_\Sigma A\right|^2 + \left|\langle A, II\rangle\right|^2} + \int_{\partial\Sigma} |A(\nu)| \geq n\left|B^n\right|^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\int_\Sigma (\det A)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}, \quad (1.2)$$

where ν is the unit outer normal vector field on $\partial\Sigma$ with respect to Σ, θ is the asymptotic volume ratio of M. Specially, when $A = fg$ for a positive smooth function f on Σ, the equality holds impling that f is constant, M is isometric to Euclidean space, and Σ is a flat round ball.

Actually, as demonstrated by the proof of Corollary 1.4, we know that inequality (1.2) is exactly the case of S. Brendle's paper in [**?**] when $A = fg$ for a positive smooth function f. Therefore, the necessity characterization of the aforementioned equation condition naturally holds.

According to S. Brendle's paper in [**?**], we know that the product $M^{n+1}\times\mathbb{R}$ and M^{n+1} have the same asymptotic volume ratio, then we can view Σ^n as a submanifold of the $(n+2)-$dimensional manifold $M\times\mathbb{R}$. Hence the inequality in Corollary 1.2 also holds in the codimension 1 setting:

Corollary 1.3. Let $(M,\bar{g})$ be a complete noncompact manifold of dimension $n+1$ with nonnegative sectional curvature. Let Σ be a compact n-dimensional submanifold of M^{n+1} with smooth boundary $\partial\Sigma$ (possibly $\partial\Sigma=\varnothing$). If A is a smooth symmetric uniformly positive definite $(0,2)-$tensor field on Σ, then

$$\int_{\Sigma}\sqrt{\left|\mathrm{div}_{\Sigma}A\right|^{2}+\left|\langle A,II\rangle\right|^{2}}+\int_{\partial\Sigma}\left|A\left(\nu\right)\right|\geq n\left|B^{n}\right|^{\frac{1}{n}}\theta^{\frac{1}{n}}\left(\int_{\Sigma}\left(\det A\right)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}},$$

where ν is the unit outer normal vector field on $\partial\Sigma$ with respect to Σ.

As applications, we can first get the following S. Brendle's Michael-Simon-Sobolev inequality in [**?**]:

Corollary 1.4. (Brendle) Let M be a complete noncompact manifold of dimension $n+m$ with nonnegative sectional curvature. Let Σ be a compact submanifold of M of dimensional n (possibly with boundary $\partial\Sigma$), and let f be a positive smooth function on Σ. If $m\geq 2$, then

$$\int_{\Sigma}\sqrt{\left|\nabla^{\Sigma}f\right|^{2}+f^{2}\left|H\right|^{2}}+\int_{\partial\Sigma}f\geq n\left[\frac{\left(n+m\right)\left|B^{n+m}\right|}{m\left|B^{m}\right|}\right]^{\frac{1}{n}}\theta^{\frac{1}{n}}\left(\int_{\Sigma}f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}},\tag{1.3}$$

where θ denotes the asymptotic volume ratio of M. Specially when $m=2$, the equality holds impling that f is constant, M is isometric to Euclidean space, and Σ is a flat round ball.

Based on Corollary 1.4 above, we get the same isoperimetric inequality on minimal submanifolds (see [**?**]). Moreover, we obtain another application concerning the nonexistence of the closed minimal submanifold, similar to the work of C. Y. Yi and Y. Zheng (see [**?**]).

Corollary 1.5. (Yi-Zheng) If $(M,\bar{g})$ is a complete noncompact Riemannian manifold of dimension $n+m\,(m\geq 2)$ with nonnegative sectional curvature and Euclidean volume growth $(\theta>0)$, then there doesn't exist any closed minimal submanifold in M.

This paper is organized as follows. In Section 2, we introduce fundamental concepts and establish a generalized trace inequality for the product of two square matrices. In Section 3, we provide the proof of Theorem 1.1. In Section 4, we present the proofs of Corollary 1.4 and Corollary 1.5.

2 Preliminaries

Let $(M,\bar{g})$ be a complete noncompact $(n+m)-$dimension manifold with nonnegative sectional curvature. Let Σ be a compact n-dimensional submanifold of M^{n+m} with smooth boundary $\partial\Sigma$ (possibly $\partial\Sigma=\varnothing$). Denote by $\bar{D}$ the Levi-Civita connection on $(M,\bar{g})$, and by $\bar{R}$ the Riemann curvature tensor of $(M,\bar{g})$. We assume that g_{Σ} is the induced Riemannian metric on Σ, D_{Σ} is the Levi-Civita connection on Σ, and ∇^{Σ} is the gradient of Σ. Let A be a smooth symmetric uniformly positive definite $(0,2)-$tensor field on Σ. For each point $x\in\Sigma$, we denote by $T_{x}\Sigma$ and $T_{x}^{\perp}\Sigma$ the tangent and normal space to Σ at x, respectively. Let $(x^{1},\ldots,x^{n})$ be a local coordinate system on Σ, the divergence of A on Σ is defined by

$$\mathrm{div}_{\Sigma}A:=g_{\Sigma}^{ki}D_{k}^{\Sigma}A_{ij}dx^{j}.$$

Let T and S be two $(0,2)$-tensor fields on Σ. In general, the inner product of T and S can be written as

$$\langle T,S\rangle=g_{\Sigma}^{ik}g_{\Sigma}^{jl}T_{ij}S_{kl}=T_{ij}S^{ij}.$$

The composition of T and S is the $(0,2)$-tensor $T \circ S$ defined by

$$(T \circ S)_{ij} = g_{\Sigma}^{kl} T_{ik} S_{lj}.$$

The det$T$ signifies the determinant of T, which is defined by the determinant of $(1,1)$-tensor $g_{\Sigma}^{ik} T_{jk} \frac{\partial}{\partial x^i} \otimes dx^j$. When $T \circ S = g_{\Sigma}$, we refer to T as the inverse tensor of S denoted by T^{-1}. Meanwhile, II denotes the second fundamental form of Σ as defined by

$$\langle II(X,Y), Z\rangle := \left\langle \bar{D}_X Y, Z \right\rangle = -\left\langle \bar{D}_X Z, Y \right\rangle,$$

where X and Y are tangent vector fields on Σ, Z is a normal vector field to Σ. Further, $\langle A, II\rangle(x)$ is the normal vector at $x \in \Sigma$ defined by

$$\langle A, II\rangle(x) = g_{\Sigma}^{ik} g_{\Sigma}^{jl} A_{ij} II(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}).$$

At last, we list the following lemma which extends the arithmetic-geometric mean inequality to the product of two square matrices version:

Lemma 2.1. (Lemma A.1 in [**?**]) For $n \in \mathbb{N}$, let A and B be square symmetric matrices of size n. Assume that A is positive definite and B is non-negative definite. Then

$$\det(AB) \leq \left(\frac{\mathrm{tr}(AB)}{n}\right)^n.$$

The equality holds if and only if $AB = \lambda I_n$ for some $\lambda \geq 0$, where I_n is the identity matrix.

3 Proof of Theorem 1.1

The primary proof of Theorem 1.1 is based on similar techniques from [**?**].

First, let's consider the case that Σ is connected. Due to the scaling invariant property of (**??**), it suffices to establish (**??**) under the following condition:

$$\int_{\Sigma} \sqrt{|\mathrm{div}_{\Sigma} A|^2 + |\langle A, II\rangle|^2} + \int_{\partial\Sigma} |A(\nu)| = n \int_{\Sigma} (\det A)^{\frac{1}{n-1}}. \tag{3.1}$$

Substituting (**??**) into (**??**), we obtain

$$n \int_{\Sigma} (\det A)^{\frac{1}{n-1}} \geq n \left[\frac{(n+m)|B^{n+m}|}{m|B^m|}\right]^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\int_{\Sigma} (\det A)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}.$$

Divide both sides of the inequality by $\left(\int_{\Sigma} (\det A)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}$ simultaneously, we have

$$\left(\int_{\Sigma} (\det A)^{\frac{1}{n-1}}\right)^{\frac{1}{n}} \geq \left[\frac{(n+m)|B^{n+m}|}{m|B^m|}\right]^{\frac{1}{n}} \theta^{\frac{1}{n}}. \tag{3.2}$$

In other words, we are left to prove (**??**) instead of (**??**) in the following proof under the condition (**??**).

Given that Σ is connected and A satisfies (**??**), we can find a classical solution $u : \Sigma \to \mathbb{R}$ to the following equation using the existence theorem of solutions for elliptic equations (see[**?**])

$$\begin{cases} \mathrm{div}_{\Sigma}\left(A\left(\nabla^{\Sigma} u\right)\right)(x) = n(\det A(x))^{\frac{1}{n-1}} - \sqrt{|\mathrm{div}_{\Sigma} A|^2(x) + |\langle A, II\rangle|^2(x)}, \text{ in } \Sigma \backslash \partial\Sigma, \\ \left\langle A\left(\nabla^{\Sigma} u\right)(x), \nu(x)\right\rangle = |A(\nu(x))|, \text{ on } \partial\Sigma. \end{cases} \tag{3.3}$$

We define

$$\Omega := \left\{x \in \Sigma \backslash \partial\Sigma : \left|\nabla^{\Sigma} u\left(x\right)\right| < 1\right\},$$
$$U := \{(x, y) : x \in \Sigma \backslash \partial\Sigma, y \in T_x^{\perp}\Sigma, \left|\nabla^{\Sigma} u\left(x\right)\right|^2 + |y|^2 < 1\}.$$

In the following, we can fix a positive number r and define a contact set

$$V_r := \left\{(\bar{x}, \bar{y}) \in U : ru\left(x\right) + \frac{1}{2} d\left(x, \exp_{\bar{x}}\left(r\nabla^{\Sigma} u\left(\bar{x}\right) + r\bar{y}\right)\right)^2 \geq ru\left(\bar{x}\right) + \frac{1}{2} r^2 \left(\left|\nabla^{\Sigma} u\left(\bar{x}\right)\right|^2 + |\bar{y}|^2\right)\right\}$$

for all $x \in \Sigma$. For each $r > 0$, we also denote a transport map $\Phi_r : T^{\perp}\Sigma \to M$ by

$$\Phi_r\left(x, y\right) = \exp_x\left(r\nabla^{\Sigma} u\left(x\right) + ry\right)$$

for all $x \in \Sigma$ and $y \in T_x^{\perp}\Sigma$. The standard elliptic regularity theory implies that the function $u \in C^{2,\gamma}(\Sigma)$ and Φ_r is of class $C^{1,\gamma}$ for each $0 < \gamma < 1$ (see[**?**]).

Along the same proof of [**?**], we need first the following key lemma:

Lemma 3.1. ([**?**], Lemma 3.4) The set

$$\{p \in M : d(x, p) < r \text{ for all } x \in \Sigma\}$$

is contained in $\Phi_r\left(V_r\right)$.

It should be noted that the set $\{p \in M : d(x, p) < r \text{ for all } x \in \Sigma\}$ is non-empty when r is large enough. Fix a point $p \in M$ with the property that $d(x, p) < r$ for all $x \in \Sigma$, we define $\varphi\left(x\right) = ru\left(x\right) + \frac{1}{2} d^2\left(x, p\right)$. To prove this Lemma 3.1, akin to the substantiation of S. Brendle, the essential aspect is that φ can reach its minimum in the interior of Σ. Due to the different boundary condition of the solution u associated with A from S. Brendle, we need to prove the following crucial proposition:

Proposition 3.2. The function $\varphi\left(x\right)$ attains its minimum in the interior of Σ.

Proof. To prove this proposition, we need to construct a curve below: For every $x_0 \in \partial\Sigma$, there exists a smooth curve $C : \tilde{\gamma} = \tilde{\gamma}\left(t\right)$, $t \in [0, \varepsilon)$ satisfying $\tilde{\gamma}\left(0\right) = x_0$, $\tilde{\gamma}'\left(0\right) = -\bar{\nu}\left(x_0\right)$ where $\bar{\nu} = \frac{A(\nu)}{|A(\nu)|}$. Since A is symmetric and positive definite, it is easy to show that $\langle \nu, \bar{\nu} \rangle = \left\langle \nu, \frac{A(\nu)}{|A(\nu)|} \right\rangle > 0$, so that $-\nu$ points to the interior of Σ, which implies the curve C moves toward the interior of Σ from $x_0 \in \partial\Sigma$. In addition, since $\left\langle A\left(\nabla^{\Sigma} u\right), \nu \right\rangle = \left\langle \nabla^{\Sigma} u, A\left(\nu\right) \right\rangle = |A\left(\nu\right)|$, then $\left\langle \nabla^{\Sigma} u, \frac{A(\nu)}{|A(\nu)|} \right\rangle = 1$, i. e. $\left\langle \nabla^{\Sigma} u, \bar{\nu} \right\rangle = 1$.

When $x_0 \in \partial\Sigma$ is a smooth point of the distance function $d\left(x, p\right)$, along the curve C we observe that:

$$\begin{aligned}
\frac{d}{dt}\left[ru\left(\tilde{\gamma}\left(t\right)\right) + \frac{1}{2} d^2\left(\tilde{\gamma}\left(t\right), p\right)\right]\Big|_{t=0} &= \left[r\left\langle \nabla^{\Sigma} u, \tilde{\gamma}'\left(t\right)\right\rangle + d\left(\tilde{\gamma}\left(t\right), p\right)\left\langle \nabla d, \tilde{\gamma}'\left(t\right)\right\rangle\right]\Big|_{t=0} \\
&= r\left\langle \nabla^{\Sigma} u, -\bar{\nu}\right\rangle + d\left(x_0, p\right)\left\langle \nabla d, -\bar{\nu}\right\rangle \\
&= -r - d\left(x_0, p\right)\left\langle \nabla d, \bar{\nu}\right\rangle \\
&\leq -r + d\left(x_0, p\right) \\
&< 0.
\end{aligned}$$

The penultimate inequality holds due to the fact that $-\left\langle \nabla d, \bar{\nu}\right\rangle \leq |\nabla d|\,|\bar{\nu}|$. Consequently, $\frac{d}{dt}\varphi\left(\tilde{\gamma}\left(t\right)\right) < 0$, implying that $\varphi\left(x\right)$ attains its minimum in the interior of Σ.

When $x_0 \in \partial\Sigma$ is not the smooth point, which is a cut point of p, the Calabi trick can be applied. Along the minimal geodesic $\bar{\gamma}$ connecting two points p and x_0, push forward the point p a little bit to a new point p_0 along $\bar{\gamma}$, then the geodesic from p_0 to x_0 should be the unique minimal geodesic satisfying $\bar{\gamma}(0) = x_0$ and $\bar{\gamma}'(0) = -\bar{\nu}$, thus $x_0 \in \partial\Sigma$ is not a cut point of p_0. So that

$$\begin{aligned}
\frac{d}{dt}\left[ru(\bar{\gamma}(t)) + \frac{1}{2}d^2(\bar{\gamma}(t), p)\right]\Big|_{t=0} &= \frac{d}{dt}\left[ru(\bar{\gamma}(t)) + \frac{1}{2}(d(\bar{\gamma}(t), p_0) + d(p_0, p))^2\right]\Big|_{t=0} \\
&= \left[r\left\langle \nabla^{\Sigma}u, \bar{\gamma}'(t)\right\rangle + (d(\bar{\gamma}(t), p_0) + d(p_0, p))\left\langle \nabla d, \bar{\gamma}'(t)\right\rangle\right]\Big|_{t=0} \\
&= r\left\langle \nabla^{\Sigma}u, -\bar{\nu}\right\rangle + d(x_0, p)\langle \nabla d, -\bar{\nu}\rangle \\
&= -r - d(x_0, p)\langle \nabla d, \bar{\nu}\rangle \\
&\leq -r + d(x_0, p) \\
&< 0.
\end{aligned}$$

Thus $\dfrac{d}{dt}\varphi(\bar{\gamma}(t)) < 0$, indicating that minimal point of $\varphi(x)$ lies in $\Sigma\backslash\partial\Sigma$. This completes the proof of Proposition 3.2. □

Proof of Lemma 3.1. Applying Proposition 3.2, we can now get Lemma 3.1 by the same techniques from S. Brendle's paper in [**?**]. □

Using Lemma 3.1, we also establish the following lemma replacing the ball with the annulus as in [**?**].

Lemma 3.3. ([**?**], Lemma 3.5) For each $0 \leq \sigma < 1$, the set

$$\{p \in M : \sigma r < d(x, p) < r \text{ for all } x \in \Sigma\}$$

is contained in the set $\Phi_r\left(\left\{(x, y) \in V_r : \left|\nabla^{\Sigma}u(x)\right|^2 + |y|^2 > \sigma^2\right\}\right)$.

To complete the proof of Theorem 1.1, we also need the following three lemmas in [**?**].

Lemma 3.4. ([**?**], Lemma 3.1) Suppose that $(\bar{x}, \bar{y}) \in V_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}\left(rt\nabla^{\Sigma}u(\bar{x}) + rt\bar{y}\right)$ for $t \in [0, 1]$. If Z is a vector field along $\bar{\gamma}$ satisfying $Z(0) \in T^{\perp}_{\bar{x}}\Sigma$ and $Z(1) = 0$, then

$$\begin{aligned}
&r\left(D^2_{\Sigma}u\right)(Z(0), Z(0)) - r\langle II(Z(0), Z(0)), \bar{y}\rangle \\
&+ \int_0^1 \left|\bar{D}_t Z(t)\right|^2 - \bar{R}(\bar{\gamma}'(t), Z(t), \bar{\gamma}'(t), Z(t))\, dt \geq 0.
\end{aligned}$$

Lemma 3.5. ([**?**], Lemma 3.2) Suppose that $(\bar{x}, \bar{y}) \in V_r$. Then $g + rD^2_{\Sigma}u(\bar{x}) - r\langle II(\bar{x}), \bar{y}\rangle \geq 0$.

Lemma 3.6. ([**?**], Lemma 3.3) Suppose that $(\bar{x}, \bar{y}) \in V_r$, and let $\bar{\gamma}(t) := \exp_{\bar{x}}\left(rt\nabla^{\Sigma}u(\bar{x}) + rt\bar{y}\right)$ for $t \in [0, 1]$. Moreover, let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $T^{\perp}_{\bar{x}}\Sigma$. Suppose that W is a Jacobi field along $\bar{\gamma}$ satisfying $W(0) \in T^{\perp}_{\bar{x}}\Sigma$ and $\left\langle \bar{D}_t W(0), e_j\right\rangle = r(D^2_{\Sigma}u)(W(0), e_j) - r\langle II(W(0), e_j), \bar{y}\rangle$ for each $1 \leq j \leq n$. If $W(\tau) = 0$ for some $0 < \tau < 1$, then W vanishes identically.

Among these, Lemma 3.5 holds when M has a nonnegative sectional curvature. Based on the above three lemmas, we also obtain the following lemma.

Lemma 3.7. ([?], Lemma 3.6) The Jacobian determinant of Φ_r satisfies

$$|\det D\Phi_r(x,y)| \le r^m \det\left(g + rD^2_\Sigma u(x) - r\langle II(x), y\rangle\right)$$

for all $(x,y) \in V_r$.

Applying Lemma 3.7, we obtain an upper bound for the Jacobian determinant of Φ_r.

Lemma 3.8. The Jacobian determinant of Φ_r satisfies

$$|\det D\Phi_r(x,y)| \le r^m \frac{1}{\det A(x)}\left(\frac{\mathrm{tr}_g A(x)}{n} + r\left(\det A(x)\right)^{\frac{1}{n-1}}\right)^n$$

for all $(x,y) \in V_r$.

Proof. Given a pair of points $(x,y) \in V_r$, by Cauchy-Schwarz inequality and $\left|\nabla^\Sigma u(x)\right|^2 + |y|^2 \le 1$, we obtain

$$\begin{aligned}
&-\left\langle \mathrm{div}_\Sigma A(x), \nabla^\Sigma u(x)\right\rangle - \langle A(x), \langle II(x), y\rangle\rangle \\
=&-\left\langle \mathrm{div}_\Sigma A(x), \nabla^\Sigma u(x)\right\rangle - \langle\langle A(x), II(x)\rangle, y\rangle \\
=&-\left\langle \mathrm{div}_\Sigma A(x) + \langle A(x), II(x)\rangle, \nabla^\Sigma u(x) + y\right\rangle \\
\le&\sqrt{|\nabla^\Sigma u(x)|^2 + |y|^2}\sqrt{|\mathrm{div}_\Sigma A|^2(x) + |\langle A, II\rangle|^2(x)} \\
\le&\sqrt{|\mathrm{div}_\Sigma A|^2(x) + |\langle A, II\rangle|^2(x)}.
\end{aligned}$$

Note that

$$\mathrm{div}_\Sigma\left(A\left(\nabla^\Sigma u\right)\right) = \left\langle \mathrm{div}_\Sigma A, \nabla^\Sigma u\right\rangle + \left\langle A, D^2_\Sigma u\right\rangle. \tag{3.4}$$

According to the equation of u, we have

$$\begin{aligned}
&\left\langle A(x), D^2_\Sigma u(x) - \langle II(x), y\rangle\right\rangle \\
=&\mathrm{div}_\Sigma\left(A\left(\nabla^\Sigma u\right)\right)(x) - \left\langle \mathrm{div}_\Sigma A(x), \nabla^\Sigma u(x)\right\rangle - \langle A(x), \langle II(x), y\rangle\rangle \\
=&n\left(\det A(x)\right)^{\frac{1}{n-1}} - \sqrt{|\mathrm{div}_\Sigma A|^2(x) + |\langle A, II\rangle|^2(x)} \\
&-\left\langle \mathrm{div}_\Sigma A(x), \nabla^\Sigma u(x)\right\rangle - \langle A(x), \langle II(x), y\rangle\rangle \\
\le&n\left(\det A(x)\right)^{\frac{1}{n-1}}.
\end{aligned}$$

Since A is symmetric positive definite, we derive by Lemma 2.1 and Lemma 3.7 that

$$\begin{aligned}
&|\det D\Phi_r(x,y)| \\
\le&r^m \det\left(g + rD^2_\Sigma u(x) - r\langle II(x), y\rangle\right) \\
=&r^m \frac{1}{\det A(x)}\det\left[A(x)\circ\left(g + rD^2_\Sigma u(x) - r\langle II(x), y\rangle\right)\right] \\
\le&r^m \frac{1}{\det A(x)}\left(\frac{\mathrm{tr}\left(A(x)\left(g + rD^2_\Sigma u(x) - r\langle II(x), y\rangle\right)\right)}{n}\right)^n \\
=&r^m \frac{1}{\det A(x)}\left(\frac{\langle A(x), g\rangle}{n} + \frac{r\langle A(x), D^2_\Sigma u(x) - \langle II(x), y\rangle\rangle}{n}\right)^n \\
\le&r^m \frac{1}{\det A(x)}\left(\frac{\mathrm{tr}_g A(x)}{n} + r\left(\det A(x)\right)^{\frac{1}{n-1}}\right)^n.
\end{aligned}$$

This completes the proof of Lemma 3.8.

□

Proof of Theorem 1.1. Given a constant σ such that $0 \leq \sigma < 1$, by Lemma 3.2 and Lemma 3.8, we conclude that

$$\begin{aligned}
&\left|\{p \in M : \sigma r < d(x,p) < r \text{ for all } x \in \Sigma\}\right| \\
\leq &\int_{\Omega} \left(\int_{\left\{y \in T_x^{\perp}\Sigma : \sigma^2 < |\nabla^{\Sigma} u(x)|^2 + |y|^2 < 1\right\}} \left|\det D\Phi_r(x,y)\right| 1_{V_r}(x,y)\, dy \right) d\mathrm{vol}(x) \\
\leq &\int_{\Omega} \left(\int_{\left\{y \in T_x^{\perp}\Sigma : \sigma^2 < |\nabla^{\Sigma} u(x)|^2 + |y|^2 < 1\right\}} r^m \frac{1}{\det A(x)} \left(\frac{\mathrm{tr}_g A(x)}{n} + r\left(\det A(x)\right)^{\frac{1}{n-1}} \right)^n dy \right) d\mathrm{vol}(x) \\
= &|B^m| \int_{\Omega} \left[\left(1 - \left|\nabla^{\Sigma} u(x)\right|^2\right)^{\frac{m}{2}} - \left(\sigma^2 - \left|\nabla^{\Sigma} u(x)\right|^2\right)_+^{\frac{m}{2}} \right] \\
&\qquad\qquad r^m \frac{1}{\det A(x)} \left(\frac{\mathrm{tr}_g A(x)}{n} + r\left(\det A(x)\right)^{\frac{1}{n-1}} \right)^n d\mathrm{vol}(x) \\
\leq &\frac{m}{2} |B^m| \left(1 - \sigma^2\right) \int_{\Omega} r^m \frac{1}{\det A(x)} \left(\frac{\mathrm{tr}_g A(x)}{n} + r\left(\det A(x)\right)^{\frac{1}{n-1}} \right)^n d\mathrm{vol}(x).
\end{aligned}$$

for all $r > 0$. Because of $m \geq 2$, the last inequality uses the mean value theorem

$$b^{\frac{m}{2}} - a^{\frac{m}{2}} \leq \frac{m}{2}(b-a)$$

for $0 \leq a \leq b \leq 1$. Dividing by r^{n+m} and letting $r \to \infty$, we have

$$\left|B^{n+m}\right| \left(1 - \sigma^{n+m}\right) \theta \leq \frac{m}{2} |B^m| \left(1 - \sigma^2\right) \int_{\Omega} \left(\det A\right)^{\frac{1}{n-1}}. \tag{3.5}$$

Next, we divide both side by $1 - \sigma$ and let $\sigma \to 1$, then

$$(n+m) \left|B^{n+m}\right| \theta \leq m |B^m| \int_{\Omega} \left(\det A\right)^{\frac{1}{n-1}} \leq m |B^m| \int_{\Sigma} \left(\det A\right)^{\frac{1}{n-1}}. \tag{3.6}$$

That is

$$\int_{\Sigma} \left(\det A\right)^{\frac{1}{n-1}} \geq \left[\frac{(n+m)\left|B^{n+m}\right|}{m |B^m|} \right] \theta.$$

This means that the above inequality is (**??**). From this, according to the beginning of the analysis, we can get (**??**). Thus, in the special case that Σ is connected, Theorem 1.1 has been proved.

It remains to consider the case when Σ is disconnected. In this case, we apply the inequality to each individual connected component of Σ, and compute the sum across all connected components. Notic that

$$a^{\frac{n-1}{n}} + b^{\frac{n-1}{n}} > a(a+b)^{\frac{1}{n}} + b(a+b)^{\frac{1}{n}} = (a+b)^{\frac{n-1}{n}}$$

for $a, b > 0$, we can conclude that

$$\int_{\Sigma} \sqrt{\left|\mathrm{div}_{\Sigma} A\right|^2 + \left|\langle A, II\rangle\right|^2} + \int_{\partial\Sigma} |A(\nu)| > n \left[\frac{(n+m)\left|B^{n+m}\right|}{m |B^m|} \right]^{\frac{1}{n}} \theta^{\frac{1}{n}} \left(\int_{\Sigma} \left(\det A\right)^{\frac{1}{n-1}} \right)^{\frac{n-1}{n}}$$

if Σ is disconnected. This completes the proof of Theorem 1.1. □

4 Proof of Corollary 1.4 And Corollary 1.5

First, let's begin the proof of Corollary 1.4.

Proof of Corollary 1.4. According to Theorem 1.1, the following inequality holds:

$$\int_{\Sigma}\sqrt{\left|\mathrm{div}_{\Sigma}A\right|^{2}+\left|\langle A,II\rangle\right|^{2}}+\int_{\partial\Sigma}\left|A\left(\nu\right)\right|\geq n\left[\frac{\left(n+m\right)\left|B^{n+m}\right|}{m\left|B^{m}\right|}\right]^{\frac{1}{n}}\theta^{\frac{1}{n}}\left(\int_{\Sigma}\left(\det A\right)^{\frac{1}{n-1}}\right)^{\frac{n-1}{n}}.$$

When $A=fg_{\Sigma}$ for a positive smooth function f on Σ, we find that

$$\mathrm{div}_{\Sigma}A=\mathrm{div}\left(fg_{\Sigma}\right)=f\mathrm{div}g_{\Sigma}+\left\langle \nabla^{\Sigma}f,g\right\rangle =\nabla^{\Sigma}f.$$

Let H denote the mean curvature vector of Σ, we have

$$\langle A,II\rangle=\langle fg_{\Sigma},II\rangle=fH.$$

It's readily apparent that $A\left(\nu\right):=A\left(\cdot,\nu\right)=fg\left(\nu\right)$. Let X be a tangent vector field on M, then $g\left(\nu\right)\left(X\right)=g\left(\nu,X\right)$, since $\left|\nu\right|=1$, it is easy to show that $\left|A\left(\nu\right)\right|=f\left|g\left(\nu\right)\right|=f$. In order to calculate the determinant of A, we first transform A from a (0, 2)-tensor field to a (1, 1)-tensor field, following this transformation, we proceed to compute the determinant of A, i. e. $A_{i}^{k}:=g^{jk}A_{ij}$, $\tilde{g}=g^{jk}g_{ij}=\delta_{i}^{k}$. Consequently, we obtain

$$\det A=\det\left(fg_{\Sigma}\right)=f^{n}\det g_{\Sigma}=f^{n}\det\tilde{g}=f^{n}.$$

Therefore, we arrive at the inequality as outlined in Theorem 1.3 of S. Brendles seminal paper in [**?**]:

$$\int_{\Sigma}\sqrt{\left|\nabla^{\Sigma}f\right|^{2}+f^{2}\left|H\right|^{2}}+\int_{\partial\Sigma}f\geq n\left[\frac{\left(n+m\right)\left|B^{n+m}\right|}{m\left|B^{m}\right|}\right]^{\frac{1}{n}}\theta^{\frac{1}{n}}\left(\int_{\Sigma}f^{\frac{n}{n-1}}\right)^{\frac{n-1}{n}}.$$

□

At last, we have the following proof of Corollary 1.5.

Proof of Corollary 1.5. The contradiction can be given. Assume that (Σ,g_{Σ}) is a closed n-dimension minimal submanifold of M. For Theorem 1.1, we can set $A=g$, and thus conclude that

$$0\geq n\left[\frac{\left(n+m\right)\left|B^{n+m}\right|}{m\left|B^{m}\right|}\right]^{\frac{1}{n}}\theta^{\frac{1}{n}}\left|\Sigma\right|^{\frac{n-1}{n}}>0.$$

It's a contradiction. This completes the proof of Corollary 1.5. □

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