On Invariants of Trigonal Fibrations

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1. Global Invariants

- Fibration $f : S \to C$ of genus g, $b := g(C)$
- Relative invariants:

$$\begin{cases} K_f^2 &= c_1^2(S) - 8(g-1)(b-1), \\ e_f &= c_2(S) - 4(g-1)(b-1), \\ \chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1) = \deg f_* \omega_{S/C}. \end{cases}$$

- **Problem**: Can the global invariants be computed by the local contributions of the fiber germs?

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- (S.-L.Tan) Generalized Kodaira formulae

$$\begin{cases} K_f^2 = \kappa(f) + \sum_F c_1^2(F), \\ e_f = \delta(f) + \sum_F c_2(F), \\ \chi_f = \lambda(f) + \sum_F \chi_F, \end{cases}$$

- Modular invariants: $\kappa(f), \delta(f), \lambda(f)$,
  Chern number: $c_1^2(F), c_2(F), \chi_F$.
- Consider this problem in semistable cases.

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2. Invariants of hyperelliptic fibrations

- (G. Xiao) If $g = 2$, then

$$K_f^2 = \frac{1}{5}s_2 + \frac{7}{5}s_3,$$
$$\chi_f = \frac{1}{10}s_2 + \frac{1}{5}s_3,$$
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- singularity counts $s_2 = \sum_F s_2(F)$, $s_3 = \sum_F s_3(F)$.
- If f is semistable, them
  $s_2(F) =$ the number of inseparable points of F
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$$K_f^2 = \frac{2}{7}s_2 + \frac{17}{7}s_3 + \frac{10}{7}s_4 + \frac{20}{7}s_5,$$
$$\chi_f = \frac{3}{28}s_2 + \frac{2}{7}s_3 + \frac{2}{7}s_4 + \frac{1}{14}s_5,$$
$$e_f = (s_2 - 2s_5) + s_3 + 2s_4,$$

- $s_k = \sum_F s_k(F)$ $(k = 2, \ldots, 5)$.
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3. Invariants of non-hyperelliptic fibrations of genus 3

- (M.Reid, G.Xiao) Conjecture: Suppose $g = 3$ and non-hyperelliptic:

$$K_f^2 = \frac{1}{3}a_0 + \frac{4}{3}a_1 + 3a_2 + \frac{14}{3}a_3,$$
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$$e_f = a_0 + a_2 + a_3,$$

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$$H_{CT}(F) := s_1(F) + 2s_2(F).$$

- Horikawa's number defined by M. Reid:

$$H_R(F) = \text{length coker } (S^2 f_* \omega_{S/C} \hookrightarrow f_*(\omega_{S/C}^{\otimes 2}))_p, \ p = f(F).$$

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4. Trigonal fibration

- Trigonal fibration: general fiber is trigonal (i.e., gonality is 3).
- (Z.-J. Chen, J. Lu, S.-T. Tan) Similar formulae are found when f is semistable.
- Method: Computation of global invariants of f
  $\Longrightarrow$ Computation of global invariants of some triple cover
  $\Longrightarrow$ Computation of local contributions of the singularities of this triple cover.
  $\Longrightarrow$ Classify the singularities of a triple cover.
- Similar method to hyperelliptic fibration (by double cover).

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1. Singularities and Finite Covers

- Study singularities by using canonical resolution
- $\pi_0 : X_0 \to Y_0$ finite cover with branch locus $R_0 := R_{\pi_0}$.

$$\begin{array}{ccccccccc} \overline{X} & \xrightarrow{\eta} & X_k & \xrightarrow{\bar{\sigma}_k} & \cdots\cdots & \longrightarrow & X_2 & \xrightarrow{\bar{\sigma}_2} & X_1 & \xrightarrow{\bar{\sigma}_1} & X_0 \\ & \searrow & \downarrow{\scriptstyle \pi_k} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0} \\ & & Y_k & \xrightarrow{\sigma_k} & \cdots\cdots & \longrightarrow & Y_2 & \xrightarrow{\sigma_2} & Y_1 & \xrightarrow{\sigma_1} & Y_0 \end{array}$$

(1) σ_{i+1} is the blowing-up of Y_i at a singular point p_i of $R_i = R_{\pi_i}$.
(2) R_k is a normal-crossing branch curve.
(3) $\overline{X} \xrightarrow{\eta} X_k$ is the resolution of the Hirzebruch-Jung singularities.

- **Problem:** Classify the singularities on X by using the information of the branch curves on Y.

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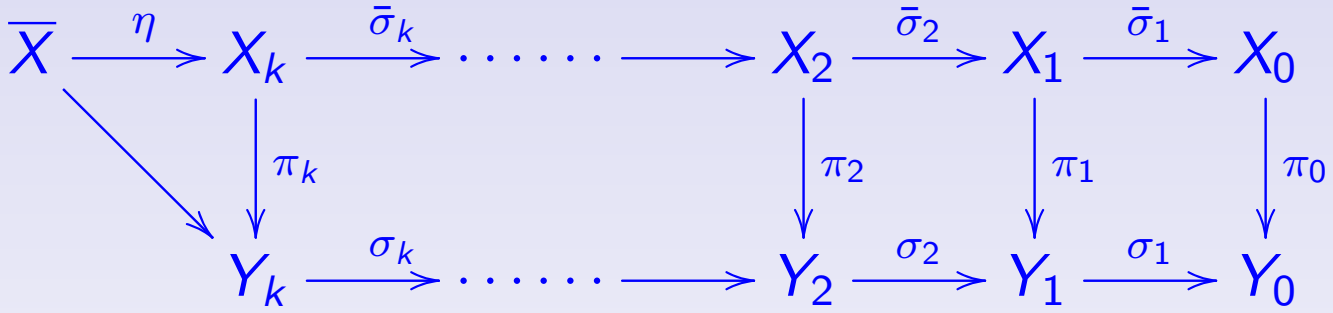
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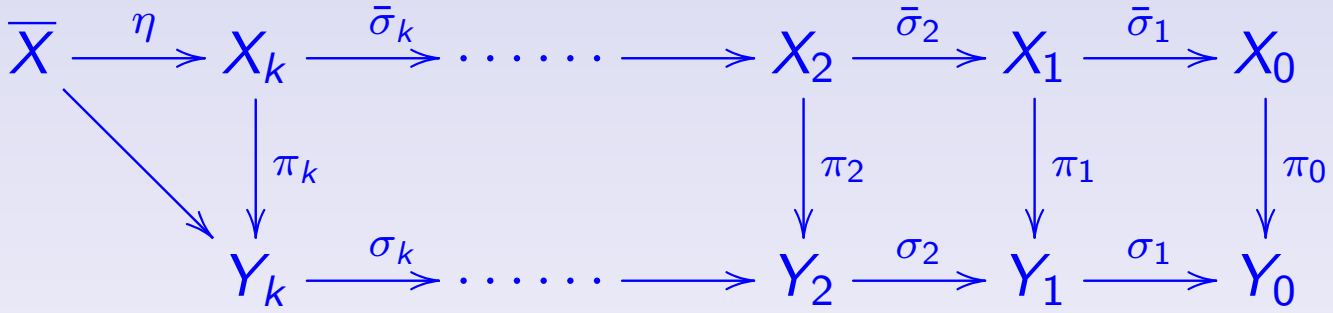


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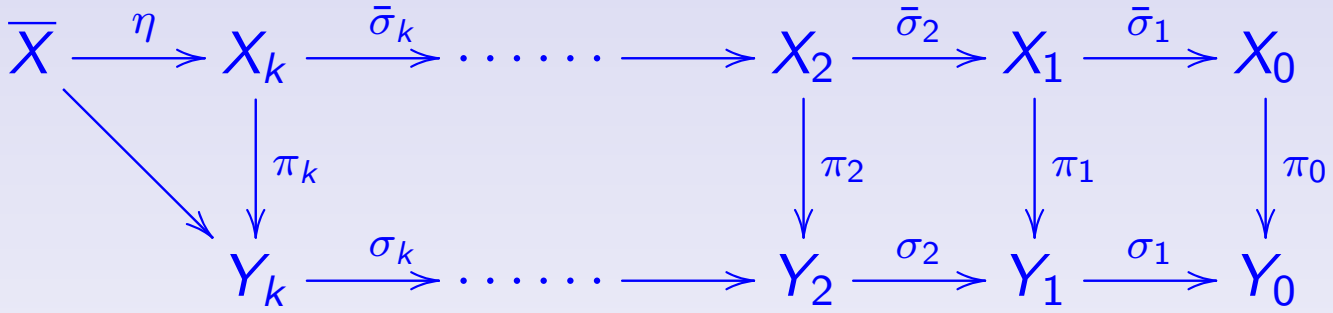


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(2) R_k is a normal-crossing branch curve.
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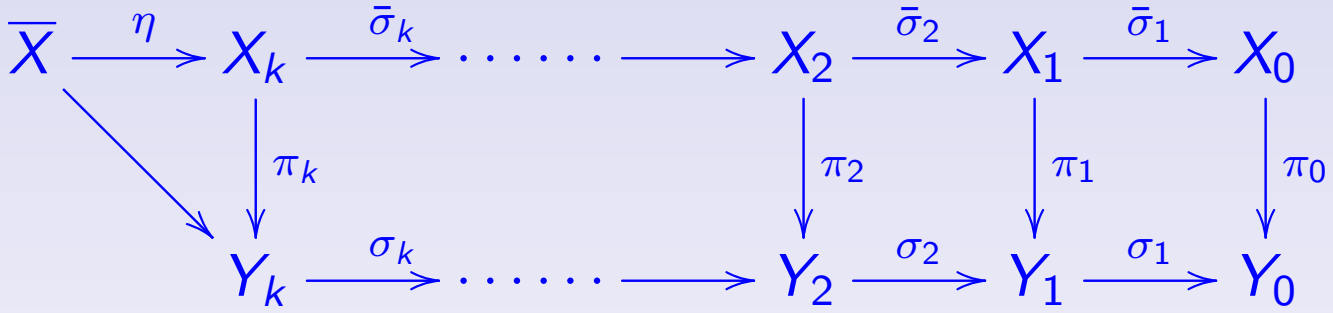


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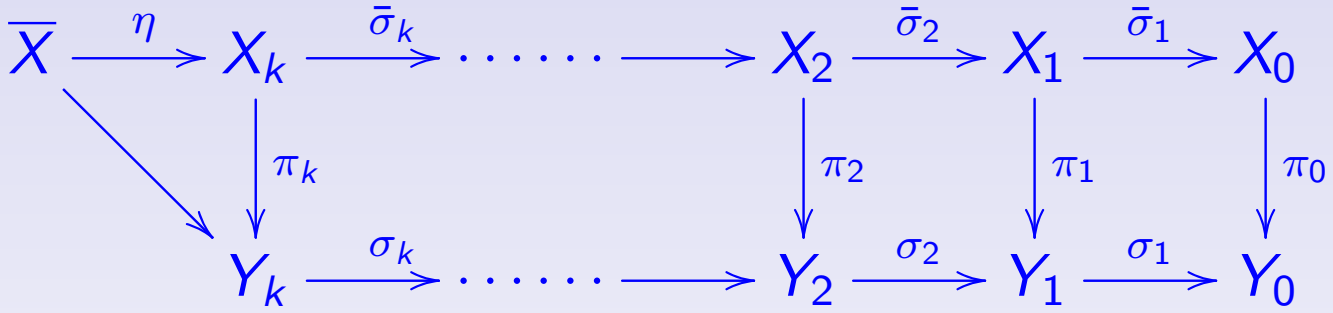


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2. Singularities of multiplicity two

- Singularities of Multiplicity Two (X_0, q_0):
- R_k is smooth, so X_k is also smooth.

$$\begin{array}{ccc} X_k & \xrightarrow{\varphi_0} & (X_0, q_0) \\ {\scriptstyle \pi_k}\downarrow{\scriptstyle 2:1} & & {\scriptstyle 2:1}\downarrow{\scriptstyle \pi_0} \\ Y_k & \xrightarrow[\psi_0]{} & (Y_0, p_0) \end{array} \qquad \begin{array}{ccc} X_k & \xrightarrow{\varphi_i} & (X_i, q_i) \\ {\scriptstyle \pi_k}\downarrow{\scriptstyle 2:1} & & {\scriptstyle 2:1}\downarrow{\scriptstyle \pi_i} \\ Y_k & \xrightarrow[\psi_i]{} & (Y_i, p_i) \end{array}$$

- $m_i = \mathrm{mult}_{p_i}(R_i), \quad w_i = \left[\frac{m_i}{2}\right], \ i = 0, 1, \cdots, k-1$
- $\mathcal{E}_i =$ the total transform on Y_k of the exceptional curve of σ_i

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- Fundamental cycle decomposition: $F_i = Z_{i,1} + Z_{i,2}$, where $Z_{i,1}$ is the fundamental cycle of the resolution $X_k \to (X_i, q_i)$, and if $Z_{i,2} \neq 0$, then $Z_{i,2} < Z_{i,1}$ and $Z_{i,1}Z_{i,2} = 0$.
- $w_i = p_a(Z_{i,1}) + p_a(Z_{i,2})$.
- (X_0, q_0) is rational $\iff w_i \leq 1$ for all $i \iff m_i \leq 3$ for all i.
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- Suppose $\deg \pi_0 = 3$.
  π_0 is determined by a minimal cubic equation $z^3 + sz + t = 0$ where
  $$s \in H^0(Y_0, \mathcal{L}^2), \quad 0 \neq t \in H^0(Y_0, \mathcal{L}^3),$$
  $\mathcal{L}$ is a line bundle.
  (Minimal: there is no prime p, s.t. $p^2 \mid s,\ p^3 \mid t$)
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  $$a = \frac{4s^3}{\gcd(s^3, t^2)}, b = \frac{27t^2}{\gcd(s^3, t^2)}, c = \frac{4s^3 + 27t^2}{\gcd(s^3, t^2)}.$$

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- Decomposition:

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where a_1, a_2, b_1, c_1 are square-free and $\gcd(a_1, a_2) = \gcd(a_i, b_j) = 1$.

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- Divisors

$$A_i = \mathrm{Div}(a_i), B_i = \mathrm{Div}(b_i), C_i = \mathrm{Div}(c_i).$$

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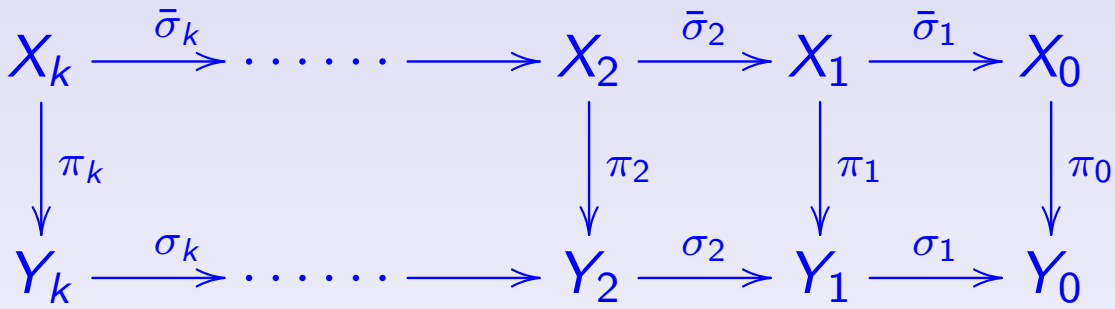
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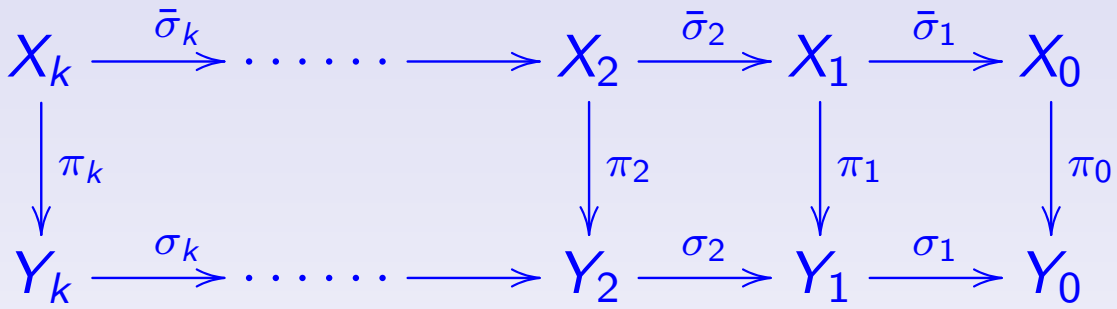
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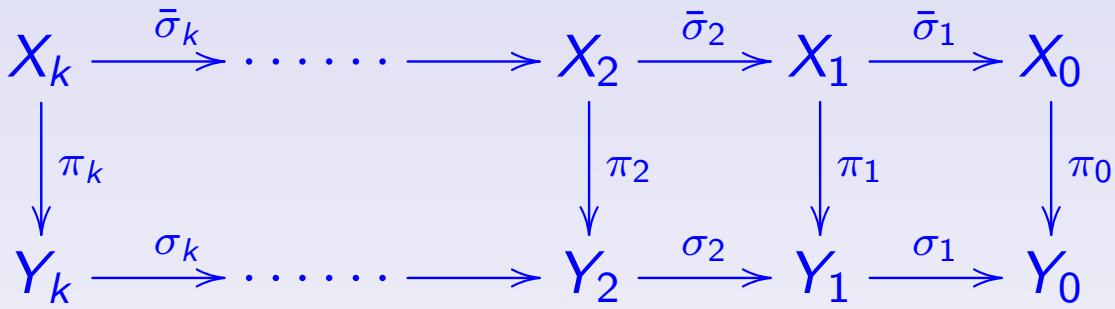
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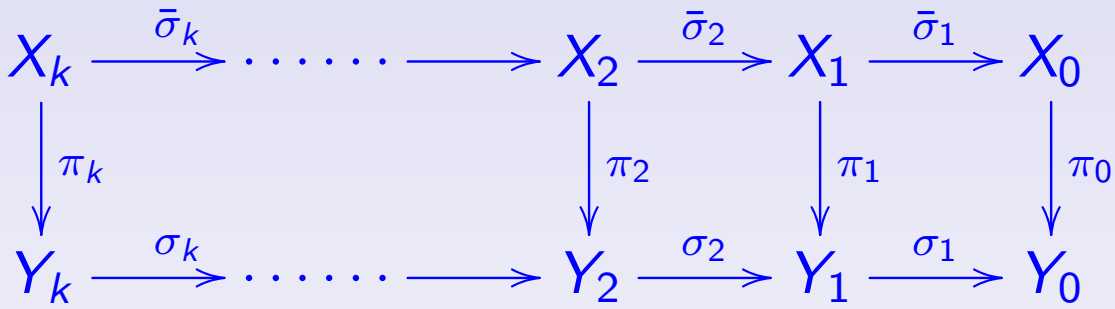
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$$\begin{array}{ccc} X_k & \xrightarrow{\varphi_0} & (X_0, q_0) \\ \pi_k \downarrow 3{:}1 & & 3{:}1 \downarrow \pi_0 \\ Y_k & \xrightarrow[\psi_0]{} & (Y_0, p_0) \end{array} \qquad \begin{array}{ccc} X_{i+1} & \xrightarrow{\bar{\sigma}_{i+1}} & (X_i, q_i) \\ \pi_i \downarrow 3{:}1 & & 3{:}1 \downarrow \pi_i \\ Y_{i+1} & \xrightarrow[\sigma_{i+1}]{} & (Y_i, p_i) \end{array}$$

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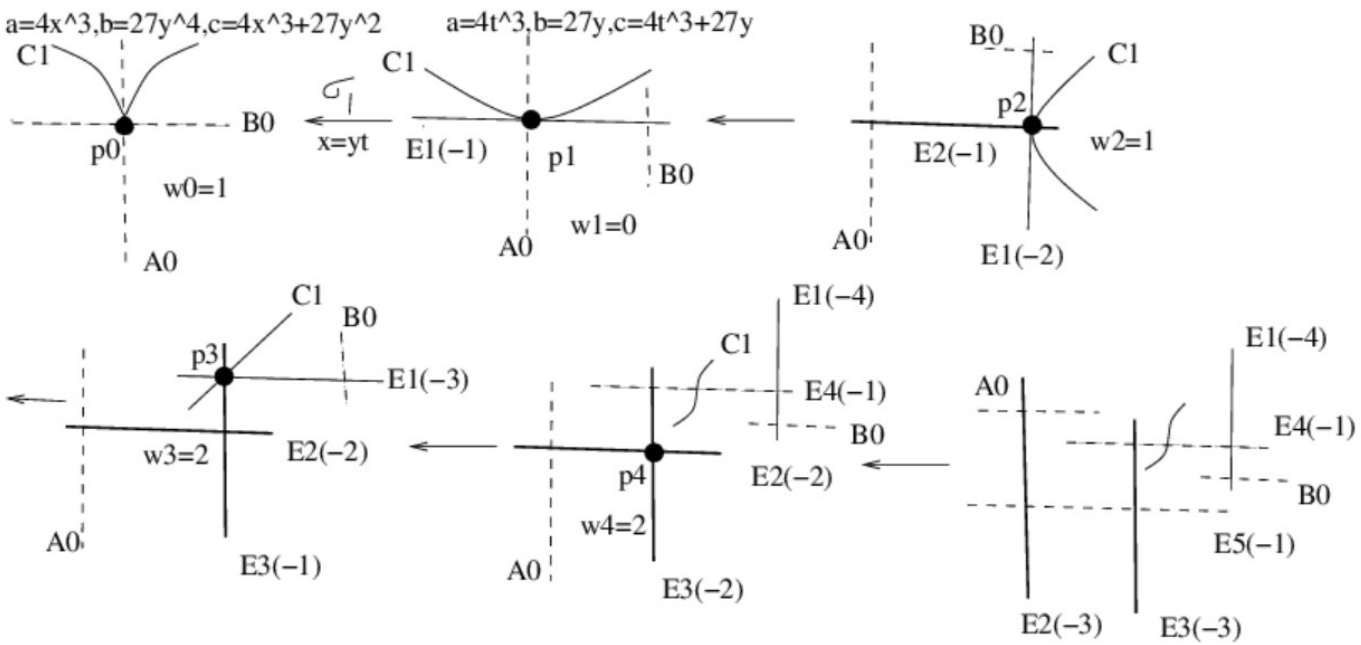
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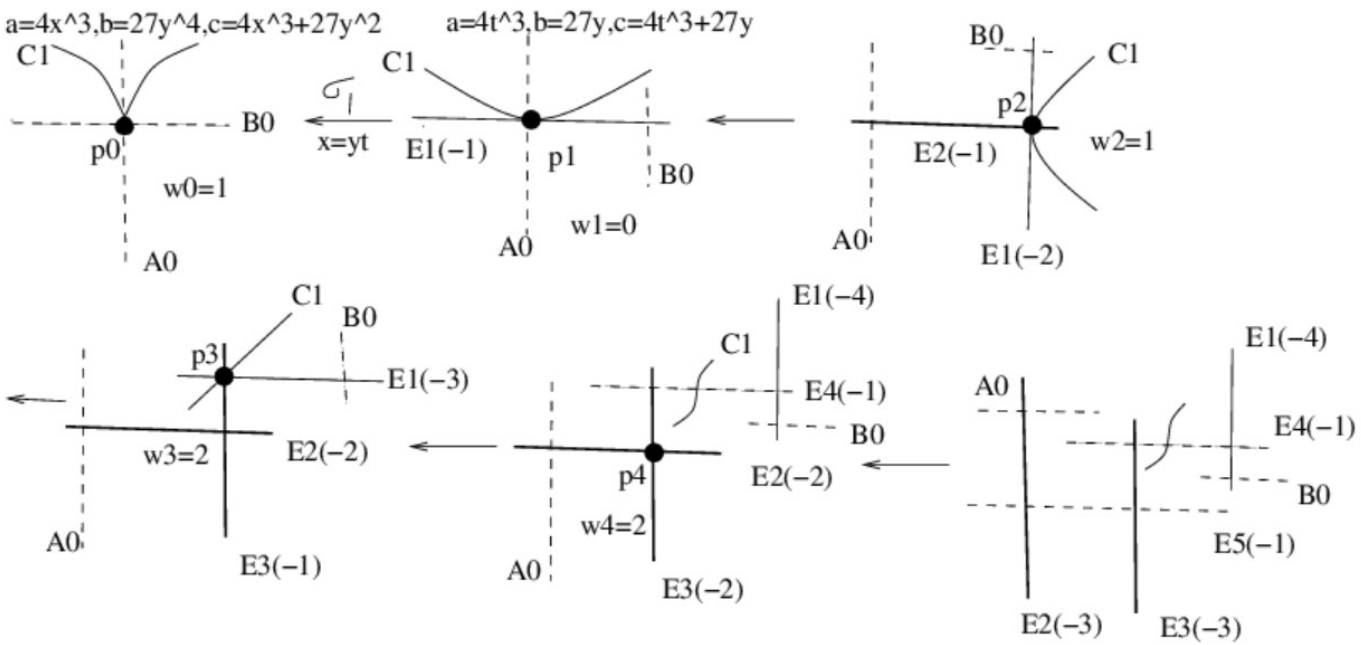
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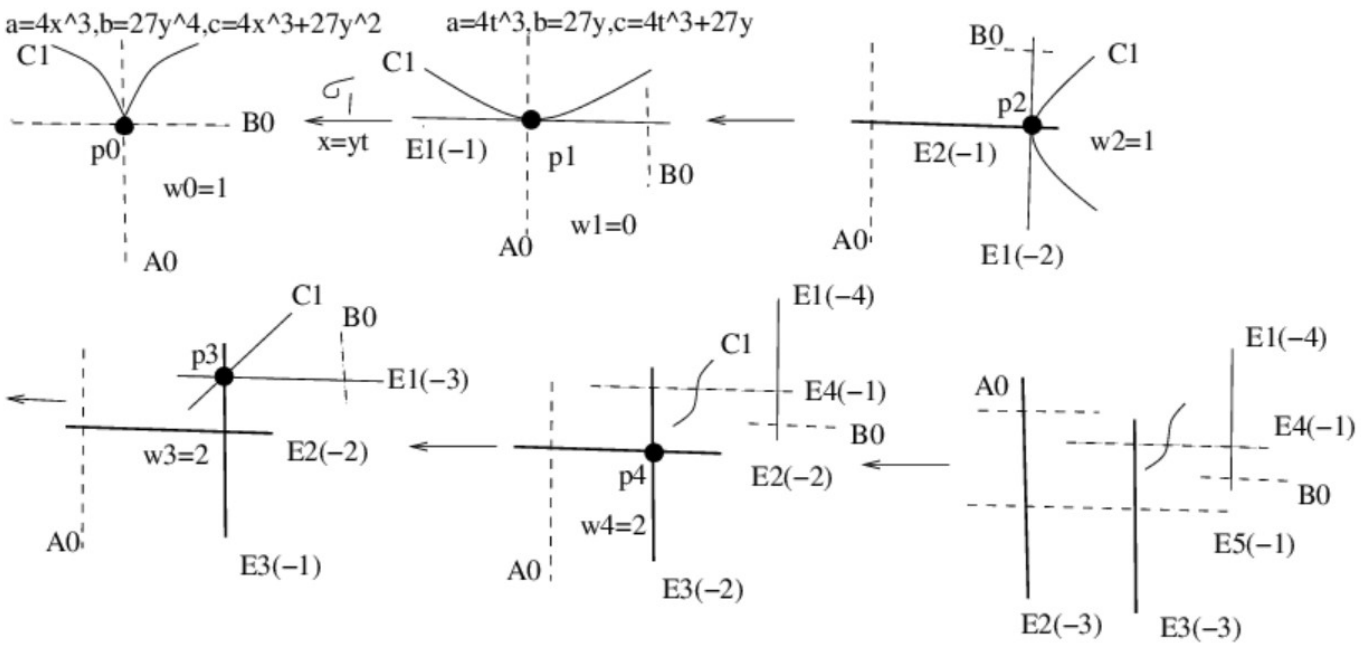
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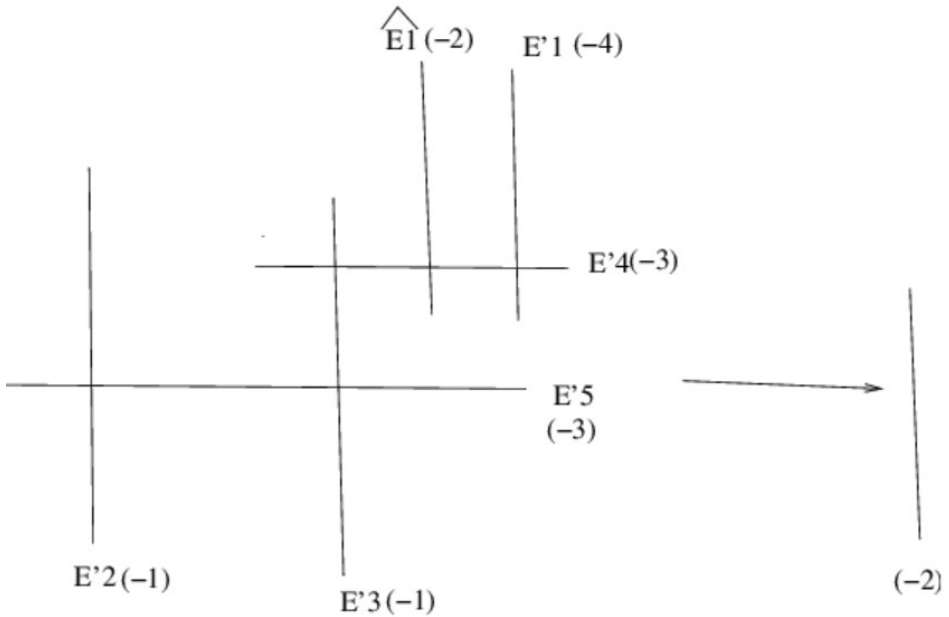
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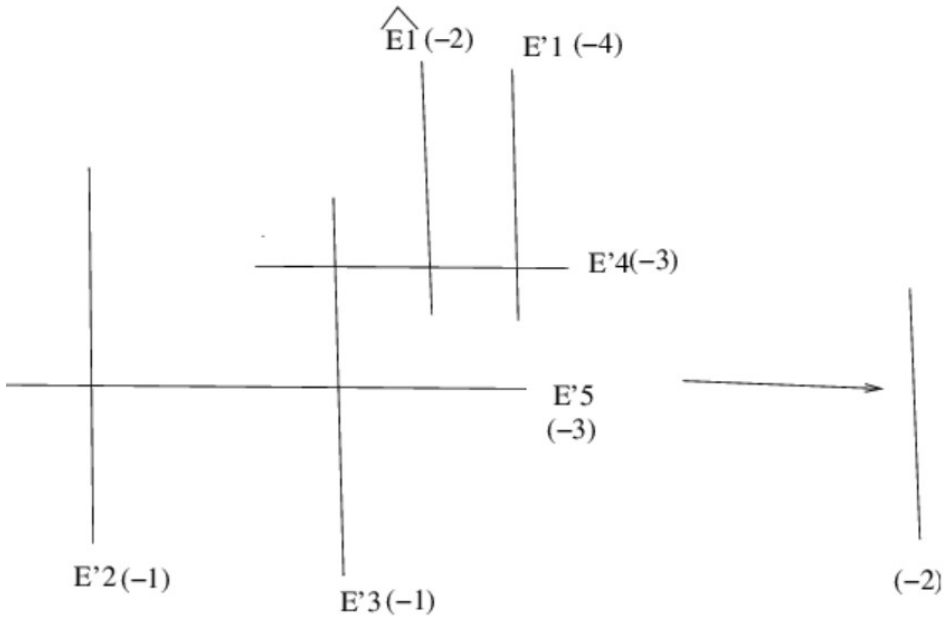
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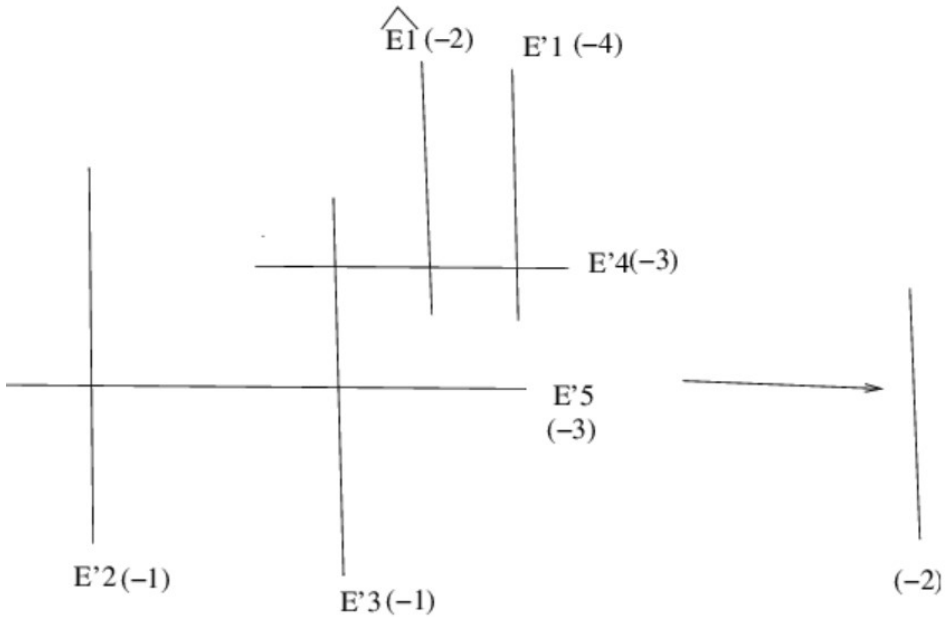
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- (S.-L. Tan) Global invariants of X_k:

$$K_{X_k}^2 = 3K_{Y_k}^2 + \frac{1}{2}D_1^2 + 2D_1K_{Y_k} + \frac{4}{3}D_2^2 + 4D_2K_{Y_k} - r_1.$$

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Suppose $\deg \pi_0 = 3$.

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  (1) D is the fundamental cycle on the exceptional curve E and $D^2 = -n$.
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1. Compute the invariants of trigonal fibration.

- Application 1: Compute the invariants of trigonal fibration.
- Method:
  - Replace f by a new fibration $\tilde{f}$ by using base changes which gives a triple cover over a ruled surface.
  - Compute the contributions of the singularities of this triple cover to the invariants of $\tilde{f}$ by canonical resolution.
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- Application 2: slop inequalities of triple fibration of type (g, γ)
- f is double cover (resp. triple cover) fibration of type (g, γ)

$\Longleftrightarrow$

(1) f relatively minimal & genus g.
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(3) $\exists$ a general double cover (resp. triple cover) $\phi : S \dashrightarrow P_0$ satisfying the following diagram commutes,

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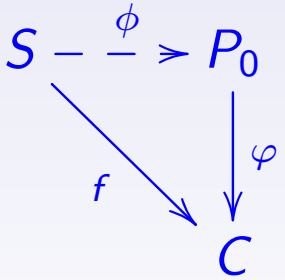
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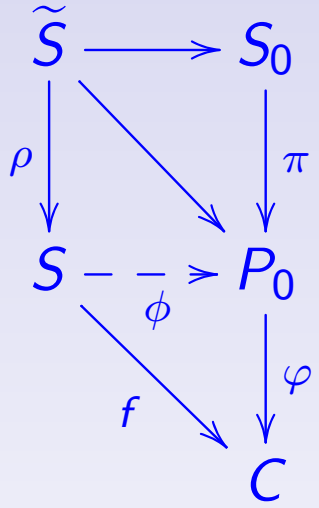
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- (P. Liu, J. Lu, F. Ye) If $g > \frac{15}{2}\gamma + \frac{5}{2}$, then

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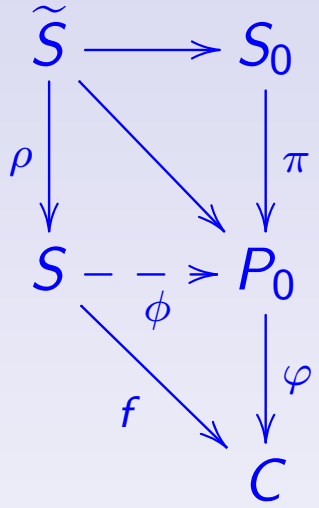


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