Final Examination for Algebraic Geometry III

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Question *Let B be a curve in $P = \mathbb{P}^1 \times \mathbb{P}^1$ defined by the equation*

$$(t-1)(t+x^{2g+2}) = 0,$$

where (t, x) is the coordinate of P and $g \geq 2$. $\pi : S_0 : toP$ be a double cover over P ramified along B.

1. *Write the above equation nearby the fiber $(t, x) = (\infty, \infty)$ by using the coordinate $(s, u) = (\frac{1}{t}, \frac{1}{x})$.*
2. *Determine the process of canonical resolution of π,*

$$\begin{array}{ccc} \widetilde{S} & \longrightarrow & S_0 \\ {\scriptstyle \tilde{\pi}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \widetilde{P} & \longrightarrow & P \end{array}$$

   where $\widetilde{S}$ is smooth.
3. *Show that the minimal model of $\widetilde{S}$ is a rational ruled surface.*
4. *Let $f : S \to \mathbb{P}^1$ be the relatively minimal fibration obtained by $\widetilde{S} \to \widetilde{P} \xrightarrow{pr_1} \mathbb{P}^1$. Compute the genus of f and invariants K_f^2, χ_f, e_f, q_f, $h^{1,1}(S)$.*
5. *Determine all singular fibers and theirs normal-crossing models.*
6. *Compute the Chern numbers of these singular fibers.*
7. *Compute the modular invariants of f.*
8. *Find a basis of $NS(S)$ and a section of f denoted by O.*
9. *For each singular fiber F_i, let Γ_i be the irreducible component of F_i which intersects O and let G_i be the intersection matrix on the irreducible components of $F_i - \Gamma_i$. Compute $|\det G_i|$*
10. *Let T be a set consisting of all irreducible components of fibers and O. We call $MW(f) := NS(S)/T$ the Mordell-Weil group of f. Show that*

$$MW(f) \cong \begin{cases} \mathbb{Z}^{\oplus 2g}, & 2 \mid g, \\ \mathbb{Z}_2 \oplus \mathbb{Z}^{\oplus 2g}, & 2 \nmid g. \end{cases}$$