On the modular invariants of a family of non-hyperelliptic curves of genus 3

Zhijie Chen, Jun Lu, Sheng-Li Tan

April 18, 2010

Abstract

We determine the modular invariants of a family of non-hyperelliptic curves of genus 3 from its special fibers. The formulas we obtained are conjectured by M. Reid.

1 Introduction

Let $f : S \to C$ be a family of complex curves of genus g, namely S is a smooth compact surface, C is a smooth curve, and the generic fiber of f is a connected smooth curve of genus g. In this paper, we always assume that S is relatively minimal, i.e., there is no (-1)-curves in any fiber. A fundamental problem is to determine the Chern numbers of S from the special fibers.

When $g = 1$, Kodaira [Kod63] found the global invariants from the singular fibers. The first Chern number $c_1^2(S)$ is always zero, the second Chern number $c_2(S)$ is equal to $12\chi(\mathcal{O}_S)$ by Noether's formula, and

$$c_2(S) = j+6\nu(\mathrm{I}^*)+2\nu(\mathrm{II})+10\nu(\mathrm{II}^*)+3\nu(\mathrm{III})+9\nu(\mathrm{III}^*)+4\nu(\mathrm{IV})+8\nu(\mathrm{IV}^*), \quad (1)$$

where $\nu(\mathrm{T})$ denotes the number of singular fibers of type T, and j is the number of poles of the J-function of the family. The modular invariant j is just the degree of the modular map J from C to the moduli space $\overline{\mathcal{M}}_1$ of semistable elliptic curves. So j depends only on the generic fibers.

Kodaira proved that one can also compute j from the singular fibers F. Let $\widetilde{F}$ be the semistable model of F under an n-cyclic local base change totally ramified over $f(F)$. We define

$$\delta(F) = \frac{1}{n}\#\{\text{ double points of } \widetilde{F}\,\}. \quad (2)$$

$\delta(F)$ is independent of the choice of base changes. Then we have

$$j = \sum_F \delta(F). \quad (3)$$

In the general case $g \geq 2$, f induces also a modular (holomorphic) map from C to the moduli space of semistable curves of genus g:

$$J : C \longrightarrow \overline{\mathcal{M}}_g\,.$$

2000 Mathematics Subject Classification. 14F05, 14H30, 13B22.
Key words and phrases. triple cover, singularity, trigonal fibration, slope.

For each class η in the Picard group of the moduli space (or stack) $\overline{\mathcal{M}}_g$, we can define an invariant $\eta(f) = \deg J^*\eta$ which satisfies the base change property, i.e., if $\widetilde{f} : \widetilde{S} \to \widetilde{C}$ is the pullback fibration of f under the base change $\pi : \widetilde{C} \to C$ of degree d, then $\eta(\widetilde{f}) = d\eta(f)$.

There are three important divisor classes λ, δ and κ on the moduli space, where λ is the Hodge divisor class, $\delta = \delta_0 + \cdots + \delta_{[g/2]}$ is the boundary divisor class and $\kappa = 12\lambda - \delta$. The modular invariants of f are just $\kappa(f)$, $\lambda(f)$, $\delta(f)$. Obviously, the modular invariants depend only on generic fibers of f. When f is semistable, they are just the following relative Chern numbers of S:

$$\begin{cases} K_f^2 = c_1^2(S) - 8(g-1)(g(C)-1), \\ \chi_f = \chi(\mathcal{O}_S) - (g-1)(g(C)-1), \\ e_f = c_2(S) - 4(g-1)(g(C)-1). \end{cases}$$

The third author generalized Kodaira's formulas to the case $g \geq 2$.

$$\begin{cases} K_f^2 = \kappa(f) + \sum_{i=1}^s c_1^2(F_i)\,, \\ \chi_f = \lambda(f) + \sum_{i=1}^s \chi_{F_i}\,, \\ e_f = \delta(f) + \sum_{i=1}^s c_2(F_i)\,, \end{cases} \tag{4}$$

where $c_1^2(F)$, $c_2(F)$ and $\chi_F = (c_1^2(F) + c_2(F))/12$ are the Chern numbers of F which can be computed locally from the combinatorial structure of F (see [Ta96] or [Ta09] for the computation formulas). If F contains no (-1)-curves, then these Chern numbers are nonnegative rational numbers, and one of them vanishes iff F is semistable. In order to find the local contributions of a fiber F to the Chern numbers of S, we only need to know the contributions to the modular invariants.

On the other hand, from the structure of the Picard group of the moduli space $\overline{\mathcal{M}}_g$, one knows that λ, κ and δ are linear combinations of some divisors corresponding to special curves. Theoretically, the modular invariants can also be computed from the special or singular fibers similar to (2) and (3).

Indeed, Horikawa [Ho77] and Xiao ([Xi85], [Xi92]) got such formulas for genus 2 fibrations.

Let $\widetilde{F}$ be the semistable model of F under an n-cyclic base change totally ramified over $f(F)$. A double point p of a semistable fiber $\widetilde{F}$ is called *separable* if $\widetilde{F}$ becomes disconnected when normalize $\widetilde{F}$ locally at p, otherwise, p is called *inseparable*. Let

$$\begin{cases} \delta_0(F) = \frac{1}{n}\#\{\,\text{inseparable double points of } \widetilde{F}\,\}, \\ \delta_1(F) = \frac{1}{n}\#\{\,\text{separable double points of } \widetilde{F}\,\}. \end{cases} \tag{5}$$

$\delta_i(F)$ is independent of the choice of the semistable reductions. If $g = 2$ and $F_1, \cdots, F_s$ are all singular fibers of f, Xiao obtained the following formulas based on Horikawa's work.

$$\begin{cases} \kappa(f) = \sum_{i=1}^s \left(\frac{1}{5}\delta_0(F_i) + \frac{7}{5}\delta_1(F_i)\right), \\ \lambda(f) = \sum_{i=1}^s \left(\frac{1}{10}\delta_0(F_i) + \frac{1}{5}\delta_1(F_i)\right), \\ \delta(f) = \sum_{i=1}^s (\delta_0(F_i) + \delta_1(F_i)). \end{cases} \tag{6}$$

Because we have $\kappa = \frac{1}{5}\delta_0 + \frac{7}{5}\delta_1$, $\lambda = \frac{1}{10}\delta_0 + \frac{1}{5}\delta_1$ and $\delta = \delta_0 + \delta_1$, The formulas (6) are equivalent to the following local-global formulas.

$$\delta_k(f) = \sum_{i=1}^{s} \delta_k(F_i), \quad k = 0, 1. \tag{7}$$

Therefore, $\delta_k(F)$ can be viewed as the local intersection number of the boundary divisor Δ_k with $J(C)$ at $J(f(F))$ in the moduli space.

By modifying the definition of $\delta_k(F)$, the formulas (5) work for the relative Chern numbers of S. Let

$$\begin{cases} s_2(F) := \delta_0(F) + 14\chi_F - 2c_1^2(F), \\ s_3(F) := \delta_1(F) + c_1^2(F) - 2\chi_F. \end{cases} \tag{8}$$

Then we have

$$\begin{cases} K_f^2 = \sum_{i=1}^{s} \left(\frac{1}{5}s_2(F_i) + \frac{7}{5}s_3(F_i)\right), \\ \chi_f = \sum_{i=1}^{s} \left(\frac{1}{10}s_2(F_i) + \frac{1}{5}s_3(F_i)\right), \\ e_f = \sum_{i=1}^{s}(s_2(F_i) + s_3(F_i)). \end{cases} \tag{9}$$

In fact, S is a generically double cover of a ruled surface, the relative invariants can be computed by using the canonical resolution of singularities. Horikawa and Xiao found the nonnegative contributions $s_3(F)$ (Horikawa number) and $s_2(F)$ respectively. The double covering method works also for hyperelliptic fibrations of genus $g \geq 3$.

The purpose of this paper is to give similar formulas for non-hyperelliptic fibrations of genus $g = 3$. Actually, M. Reid has given some conjectured formulas which relate the relative invariants with the contributions of singular and hyperelliptic fibers. His method is to decompose a special fiber F into some atomic fibers so that the contribution of F to the global invariants is the sum of the contributions of the atomic fibers. In order to explain these contributions more geometrically, we still use the notations of some divisor classes in the moduli space (or stack) $\overline{\mathcal{M}}_3$. Let h be the divisor class corresponding to the divisor of hyperelliptic locus. $\delta_k(F)$ is defined as above.

The following relations between the divisor classes are well-known.

$$\kappa = \frac{1}{3}\delta_0 + 3\delta_1 + \frac{4}{3}h, \qquad \lambda = \frac{1}{9}\delta_0 + \frac{1}{3}\delta_1 + \frac{1}{9}h, \qquad \delta = \delta_0 + \delta_1.$$

So

$$\begin{cases} \kappa(f) = \frac{1}{3}\delta_0(f) + 3\delta_1(f) + \frac{4}{3}h(f), \\ \lambda(f) = \frac{1}{9}\delta_0(f) + \frac{1}{3}\delta_1(f) + \frac{1}{9}h(f), \\ \delta(f) = \delta_0(f) + \delta_1(f). \end{cases}$$

We only need to compute $\delta_0(f)$, $\delta_1(f)$ and $h(f)$ from the special fibers.

Theorem 1.1. *Let $f : S \to C$ be a family of non-hyperelliptic curves of genus $g = 3$, and let $\delta_0(F)$ and $\delta_1(F)$ be defined as in* (5)*. Then*

$$\delta_k(f) = \sum_{i=1}^{s} \delta_k(F_i), \quad k = 0, 1.$$

Our next purpose is to find a formula to compute $h(F)$ so that $h(f) = \sum_F h(F)$. We see that $\delta_k(F)$ depends only on the combinatorial structure of F, but $h(F)$ depends also on the neighborhood structure.

When F is smooth, $h(F)$ is just the *Horikawa number* H_F defined by Reid. $h(F) = 0$ iff F is non-hyperelliptic.

In general, the Horikawa number H_F is defined as follows.

$$H_F := \text{length coker } (\mathrm{Sym}^2 f_*\omega_{S/C} \hookrightarrow f_*(\omega_{S/C}^{\otimes 2}))_p, \quad p = f(F).$$

Let $\widetilde{F}$ be the semistable model of F under an n-cyclic base change totally ramified over $f(F)$. We define the *modular Horikawa number*

$$H(F) := \frac{H_{\widetilde{F}}}{n}. \tag{10}$$

This number is independent of the choice of the semistable reductions, and it satisfies the base change property: if F' is the pullback fiber of F under a base change of degree d totally ramified over $f(F)$, then $H(F') = dH(F)$. In [Ta96], the following equality is proved,

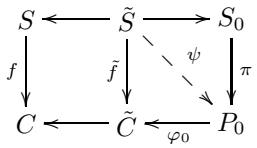
$$H(F) = H_F - \frac{3c_1^2(F) - c_2(F)}{4}. \tag{11}$$

Now we have

Theorem 1.2. *Let $f : S \to C$ be a family of non-hyperelliptic curves of genus $g = 3$, and let $h(F) = H(F) - 2\delta_1(F)$. Then $h(F) \geq 0$, and $h(F) = 0$ iff the semistable model $\widetilde{F}$ of F is non-hyperelliptic. In particular, $h(f) = \sum_F h(F)$.*

2 Computation of $\kappa(f)$, $\lambda(f)$ and $\delta(f)$

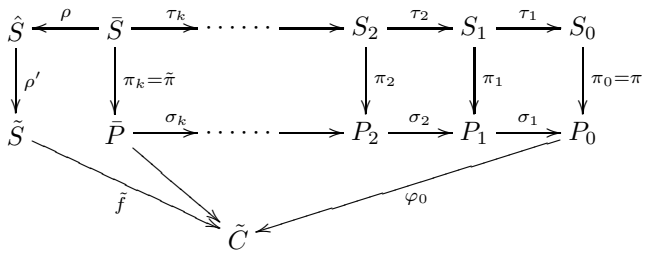
Let $f : S \to C$ be a relatively minimal non-hyperelliptic fibration of genus 3. After some base change, one can get a relatively minimal semistable fibration $\tilde{f} : \tilde{S} \to \tilde{C}$ which admits a section Γ. Thus we get a generic triple cover $\psi : \tilde{S} \dashrightarrow P_0 = \mathbb{P}(\tilde{f}_*\omega_{\tilde{S}/\tilde{C}}(-\Gamma))$ where $\varphi_0 : P_0 \to \tilde{C}$ is a ruled surface. By Stein factorization, one has a triple cover $\pi : S_0 \to P_0$.



By the base change property, $\kappa(f)$ (Resp. $\lambda(f)$, $\delta(f)$) are just the multiplicity of $\kappa(\tilde{f})$ (Resp. $\lambda(\tilde{f})$, $\delta(\tilde{f})$). So one only need to compute the latter. Our first goal is to compute $c_1^2(\tilde{S})$, $c_2(\tilde{S})$ and $\chi(\mathbb{O}_{\tilde{S}})$. by using the theory of triple cover. The details of triple covers can be referred to Sec. ?? or [Ta02].

2.1 Global invariants of triple covers.

We consider the canonical resolution of π



(12)

where σ_{i+1} is the blowing-up of P_i at a singular point p_i of the branch locus of π_i and S_{i+1} is the normalization of $P_{i+1} \times_{P_i} S_i$. $\bar{\pi} = \pi_k$ has a smooth branch locus. So $\bar{S} = S_k$ is smooth. ρ contracts all (-1)-curves in the exceptional set of the canonical resolution. ρ' contracts other (-1)-curves in the fibers.

Let D_1 (D_2) be the simple (totally) ramified branch locus of $\pi : S_0 \to P_0$. [Ta02] gives the formulae for global invariants of $\bar{S}$ as follows.

$$\begin{aligned}\chi(\mathcal{O}_{\bar{S}}) = 3\chi(\mathcal{O}_{P_0}) + \frac{1}{8}D_1^2 + \frac{1}{4}D_1K_{P_0} + \frac{5}{18}D_2^2 + \frac{1}{2}D_2K_{P_0}\\ - \sum_{i=0}^{k-1}\frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1}\frac{n_i(5n_i-9)}{18},\end{aligned} \tag{13}$$

$$\begin{aligned}K_{\bar{S}}^2 = 3K_{P_0}^2 + \frac{1}{2}D_1^2 + 2D_1K_{P_0} + \frac{4}{3}D_2^2 + 4D_2K_{P_0}\\ - \sum_{i=0}^{k-1}2(m_i-1)^2 - \sum_{i=0}^{k-1}\frac{4n_i(n_i-3)}{3} - k.\end{aligned} \tag{14}$$

where m_i and n_i are some invariants of the multiplicities of the branch locus of π_i at p_i. Their definitions can be seen in Sec.??.

2.2 Relative ramification index.

Now we will modify the above equalities (13) and (14).

Recall the relative ramification index of an effective divisor D in the ruled surface $\varphi_0 : P_0 \to \tilde{C}$

$$r_D = D(D + K_{P_0/\tilde{C}}).$$

Suppose that D be a horizontal reduced effective divisor (i.e. D does not contain any fibres of φ_0). If after k blow-ups σ_i the strict transform $\tilde{D}$ of D becomes smooth (it may be composed of several disjoint nonsingular curves), then

$$r_D = 2\sum_{p\in D}\delta_p(D) + \{\text{ramification index of this finite morphism } \tilde{D} \to \tilde{C}\},$$

where $\sum_{p\in D}$ runs over all singular points of D, and $\delta_p(D)$ is the geometric genus of $p \in D$.

Noting that $dim_{\mathbb{Q}} NS(P_0) = 2$, one has easily

$$\begin{aligned} 0 =&(F_0D)r_D - D^2(F_0D-1),\\ 0 =&r_D + (K_{P_0/\tilde{C}}D)(F_0D-1),\\ 0 =&(K_{P_0/\tilde{C}}D')(F_0D) + (K_{P_0/\tilde{C}}D)(F_0D') + 2DD', \end{aligned} \tag{15}$$

where F_0 is a fiber of φ_0 and D' is another effective divisor. Applying the above equalities to (13) and (14), we have

$$\begin{aligned} K_{\tilde{f}}^2 =&\frac{1}{3}r_{D_1} + \frac{2}{3}r_{D_2} - \frac{2}{3}D_1D_2 - \sum_{i=0}^{k-1}(2(m_i-1)^2 + \sum_{i=0}^{k-1}\frac{4n_i(n_i-3)}{3} + 1) + \varepsilon,\\ \chi_{\tilde{f}} =&\frac{1}{9}r_{D_1} + \frac{2}{9}r_{D_2} - \frac{1}{18}D_1D_2 - \sum_{i=0}^{k-1}\frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1}\frac{n_i(5n_i-9)}{18},\\ e_{\tilde{f}} =&r_{D_1} + 2r_{D_2} - \sum_{i=0}^{k}(2m_i(2m_i-1) + 2n_i(n_i-1) - 3) - \varepsilon, \end{aligned} \tag{16}$$

where ε is the number of (-1)-curves contracted by ρ and ρ', that is, $\varepsilon = \chi_{top}(\bar{S}) - \chi_{top}(\tilde{S})$.

Recall the canonical resolution diagram (12). Let E_i be the exceptional curve of σ_i, and $\mathcal{E}_i$ be the totally transform of E_i in $\bar{P}$. Let $\bar{D}_1$ ($\bar{D}_2$) be the simple (totally) ramified branch locus of $\tilde{\pi} : \bar{S} \to \bar{P}$. Let $\sigma = \sigma_1 \cdots \sigma_k : \bar{P} \to P_0$. Since

$$\begin{aligned} \bar{D}_1 &= \sigma^*(D_1) - 2\sum_{i=0}^{k-1} m_i\mathcal{E}_{i+1},\\ \bar{D}_2 &= \sigma^*(D_2) - \sum_{i=0}^{k-1} n_i\mathcal{E}_{i+1}, \end{aligned} \tag{17}$$

we have

$$\begin{aligned} r_{D_1} =&\sum_{i=0}^{k-1} 2m_i(2m_i-1) + r_{\bar{D}_1},\\ r_{D_2} =&\sum_{i=0}^{k-1}(n_i-1)n_i + r_{\bar{D}_2},\\ D_1D_2 =&\sum_{i=0}^{k-1} 2m_in_i. \end{aligned} \tag{18}$$

Combining the above equalities and (19), we obtain the following formulae for the relative invariants of $\tilde{f} : \tilde{S} \to \tilde{C}$.

$$\begin{aligned} K_{\tilde{f}}^2 =&\frac{1}{3}r_{\bar{D}_1} + \frac{2}{3}r_{\bar{D}_2} - \sum_{i=0}^{k-1}(\frac{2}{3}w_i^2 - \frac{10}{3}w_i + 3) + \varepsilon,\\ \chi_{\tilde{f}} =&\frac{1}{9}r_{\bar{D}_1} + \frac{2}{9}r_{\bar{D}_2} - \sum_{i=0}^{k-1}(\frac{1}{18}w_i^2 - \frac{5}{18}w_i),\\ e_{\tilde{f}} =&r_{\bar{D}_1} + 2r_{\bar{D}_2} + 3k - \varepsilon, \end{aligned} \tag{19}$$

where $w_i = m_i + n_i$.

2.3 Local computation of modular invariants.

Let $R = D_1 + 2D_2$ be the branch locus of $\pi : S_0 \to P_0$. After some base changes, we can assume R is composed of sections. Without a loss of generality, we can assume $\tilde{f} : \tilde{S} \to \tilde{C}$ satisfies the condition in what follows. Therefore $\bar{R} = \bar{D}_1 + 2\bar{D}_2$ lies in fibers of the ruled surface $\bar{P} \to \tilde{C}$. Recall the canonical resolution diagram (12). Let $E_{i,k}$ be the strict transform of E_i in $\bar{P}$. Thus

$$\begin{aligned} r_{\bar{D}_1} &= -2 \sum_{E_{i,k} \subseteq \bar{D}_1} 1, \\ r_{\bar{D}_2} &= -2 \sum_{E_{i,k} \subseteq \bar{D}_2} 1. \end{aligned}$$

Let F be a fiber of $\tilde{f} : \tilde{S} \to \tilde{C}$ and F_0 be the image fiber in $\varphi_0 : P_0 \to \tilde{C}$. Let $\bar{F}_0$ be the inverse image fiber of F_0 in $\bar{P}$. Put

$$\eta(F) = \sum_{E_{i,k} \subseteq \bar{D}_1 \cap \bar{F}_0} 1 + 2 \sum_{E_{i,k} \subseteq \bar{D}_2 \cap \bar{F}_0} 1.$$

We define some local invariants of F as follows.

$$\begin{aligned} \kappa(F) &= -\frac{2}{3}\eta(F) - \sum_{E_{i,k} \in \bar{F}_0} (\frac{2}{3}w_i^2 - \frac{10}{3}w_i + 3) + \varepsilon(F), \\ \lambda(F) &= -\frac{2}{9}\eta(F) - \sum_{E_{i,k} \in \bar{F}_0} (\frac{1}{18}w_i^2 - \frac{5}{18}w_i), \\ \delta(F) &= -2\eta(F) + 3 \sum_{E_{i,k} \in \bar{F}_0} 1 - \varepsilon(F), \end{aligned} \tag{20}$$

where $\varepsilon(F)$ is the contribution of F to ε.

From (19), we have

$$\begin{aligned} \kappa(\tilde{f}) &= \sum_F \kappa(F), \\ \lambda(\tilde{f}) &= \sum_F \lambda(F), \\ \delta(\tilde{f}) &= \sum_F \delta(F), \end{aligned} \tag{21}$$

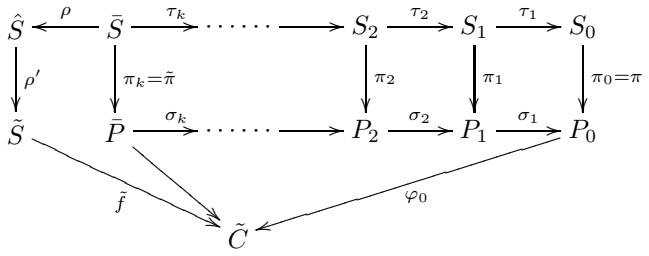
where F runs over all singular fibers.

Furthermore, we define $H(F) = \kappa(F) - 3\lambda(F)$ as Horikawa's number of F. It is easy to see that

$$H(F) = \frac{1}{2} \sum_{E_{i,k} \in \bar{F}_0} (3 - w_i)(w_i - 2) + \varepsilon(F). \tag{22}$$

3 Local computation of $\kappa(F)$, $\lambda(F)$ and $\delta(F)$

We adopt all notations and assumptions in the above sections. Recall the triple cover $\pi : S_0 \to P_0$ and the canonical resolution.



We denote $E_{i,j}$ as the strict transform of E_i in P_j $(j \geq i)$. Let $\mathcal{E}_i$ be the totally transform of E_i in $\bar{P}$.

Let F be a fiber of $\tilde{f} : \tilde{S} \to \tilde{C}$ and $\bar{F}$ (Resp. $\bar{F}_0$, F_0)be the image fiber in $\bar{S}$ (Resp. $\bar{P}$, P_0). Let $\bar{F} = \sum_{\Gamma} \eta_\Gamma \Gamma$ where Γ runs over all irreducible components of $\bar{F}$ and η_Γ is the multiplicity of Γ in $\bar{F}$. Similarly we can write $\bar{F}_0 = \sum_{\Gamma} \eta_C C$.

Let $\varepsilon(F) = \varepsilon'(F) + \varepsilon''(F)$ where $\varepsilon'(F)$ (Resp. $\varepsilon''(F)$) is the number of (-1)-curves of contracted by ρ (Resp. ρ')in $\bar{F}$.

By the definition of $\kappa(F)$, $\lambda(F)$ and $\delta(F)$, they are due to the contributions of singular points of branch locus in F_0. Hence we need classify singular points of branch locus firstly.

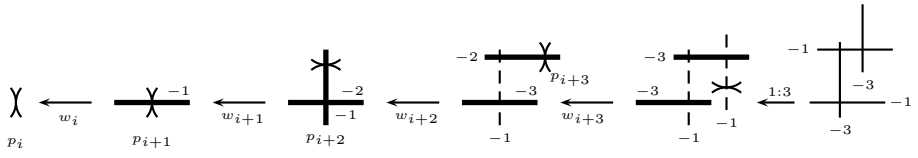
3.1 Classification of singular points of branch locus.

In this section, we will classify all singular points of branch locus from the canonical resolution. To illustrate the canonical resolution of singularities, thick line is used to represent the component contained in the triple ramification divisor D_2. Thin line is used to represent the component contained in the double ramification divisor D_1, and dashed line is used to represent the component not contained in the branch locus. The self-intersection number is marked near the component. The pair of numbers under an arrow $\underset{w_i}{\longleftarrow}$ represents the invariants w_i associated with this blowing-up. And $\overset{1:3}{\longleftarrow}$ represents a triple cover. w_i is the resolution data of p_i defined as in Sec.??

We will state the following proposition. A proof will be given in Sec. ??

Lemma 3.1. *Each singular point p of branch locus (including its infinitely close singular points occurring in the canonical resolution) is one of the following types:*

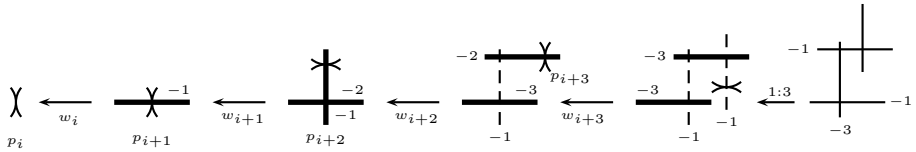
Type I_0 : *p_i is a totally ramified good cusp.*



where $w_i = 0$, $w_{i+1} = 1$, $w_{i+2} = w_{i+3} = 2$.

In what follows, p_i *as well as its infinitely close singular points* p_{i+1}, p_{i+2}, p_{i+3} *is regarded as one singular point, also denoted by* p_i. *Then* p_i *contributes* 4 *contractible curves in the exceptional set of the above resolution.*

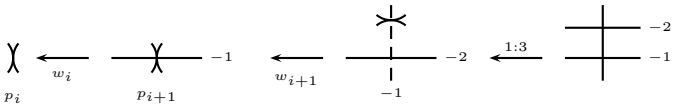
Type I_w**:** p_i *is totally ramified*



where $w_i = w$, $w_{i+1} = w + 1$, $w_{i+2} = 2$, $w_{i+3} = w + 2$.

p_i *as well as its infinitely close singular points* p_{i+1}, p_{i+2}, p_{i+3} *is regarded as one singular point, also denoted by* p_i. *Then* p_i *contributes* 3 *contractible curves in the exceptional set of the above resolution.*

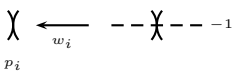
Type II_w**:** p_i *is totally ramified and* E_{i+1} *is simply ramified.*



where $w_i = w$, $w_{i+1} = w + 1$.

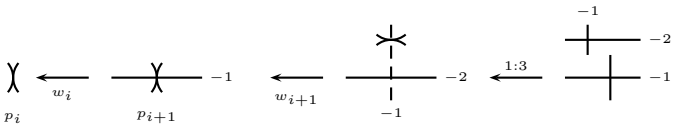
p_i *as well as its infinitely close singular points* p_{i+1} *is regarded as one singular point, also denoted by* p_i. *Then* p_i *contributes only one contractible curves in the exceptional set of the above resolution.*

Type III_w**:** p_i *is totally ramified.*



where $w_i = w$. p_i *contributes no contractible curve in the exceptional set of the above resolution.*

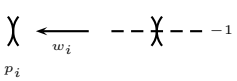
Type I_w^***:** p *is simply ramified and* E_{i+1} *is a simple ramified component.*



where $w_i = w$, $w_{i+1} = w + 1$.

p_i *as well as its infinitely close singular points* p_{i+1} *is regarded as one singular point, also denoted by* p_i. *Then* p_i *contributes* 3 *contractible curves in the exceptional set of the above resolution*

Type II_w^***:** p_i *is simply ramified.*



where $w_i = w$. p_i *contributes only one contractible curve in the exceptional set of the above resolution.*

Furthermore, any curves contracted by ρ *come from the contributions of the above singular points.*

In fact, the singular point of type I_w^* (resp. type II_w^*) is just $(2w+1 \to 2w+1)$ singularity (resp. singularity of order $2w$). The details can be referred to [Xi85, Xi92].

In view of the deformation of singular points, the above result says any singular points can split into some of the five types of singularities.

3.2 Singularity counts.

Recall $\bar{F}$ is the totally transform of a fiber F in $\bar{S}$ and $\varepsilon'(F)$ is the number of (-1)-curves of contracted by ρ in $\bar{F}$. The singular count, denoted by $s(T,F)$, is the number of singular points of type T in $\bar{F}$.

Each curve contracted by ρ' in figure (12) is called as a extra contractible curve. The singularity counts $s(T,F)$ can be written as

$$s(T,F) = s'(T,F) + s''(T,F),$$

where $s''(T,F)$ is the number of singular points of type T contributing to the extra contraction $\varepsilon''(F)$.

3.3 The proofs of Theorem 1.1 and 1.2.

We have the following lemma whose proof can be referred in Sec.??

Lemma 3.2. *Recall the local modular invariants $\delta_0(F)$ and $\delta_1(F)$ defined as in (5). We have*

$$\begin{aligned}
\varepsilon'(F) =& 4s(I_0,F) + 3(s(I_1,F) + s(I_2,F) + s(I_1^*,F) + s(I_2^*,F)) \\
&+ s(II_1,F) + s(II_2,F) + s(II_1^*,F) + s(II_2^*,F), \qquad (23)\\
\varepsilon''(F) =& 3s''(II_1,F) + 2s''(II_2,F) + s''(III_2,F) + 2s''(II_1^*) + s''(II_2^*), \qquad (24)\\
\delta_0(F) =& 3s(II_1,F) + 3s(II_2,F) + 3s'(III_2,F) \\
&+ 2s''(III_2,F) + 2s(II_1^*,F) + 2s'(II_2^*,F)), \qquad (25)\\
\delta_1(F) =& s(I_1,F) + s(I_2,F) + s''(II_2,F) \\
&+ s(I_1^*,F) + s(I_2^*,F) + s''(II_2^*,F). \qquad (26)
\end{aligned}$$

Take

$$h(F) = 3s''(II_1,F) + s(II_2) + s''(III_2) + s(I_2^*) + 2s''(II_1^*) + s'(II_2^*).$$

Combining (20) and Lemma3.2, one gets

$$\begin{aligned}
\kappa(F) &= \frac{1}{3}\delta_0(F) + 3\delta_1(F) + \frac{4}{3}h(F),\\
\lambda(F) &= \frac{1}{9}\delta_0(F) + \frac{1}{3}\delta_1(F) + \frac{1}{9}h(F),\\
\delta(F) &= \delta_0(F) + \delta_1(F),\\
H(F) &= h(F) + 2\delta_1(F).
\end{aligned}$$

Therefore we obtain the formulae for modular invariants $\kappa(f)$,$\lambda(f)$,$\delta(f)$ again. Namely,

$$\kappa(f) = \frac{1}{3}\delta_0(f) + 3\delta_1(f) + \frac{4}{3}h(f), \tag{27}$$

$$\lambda(f) = \frac{1}{9}\delta_0(f) + \frac{1}{3}\delta_1(f) + \frac{1}{9}h(f), \tag{28}$$

$$\delta(f) = \delta_0(f) + \delta_1(f), \tag{29}$$

where

$$h(f) = \sum_F h(F).$$

can also be regarded as the sum of local contributions of singular fibers to the modular invariant $h(f)$ corresponding to a hyperelliptic divisor in modular space $\mathcal{M}_3$.

From the well-known formulae for modular invariants of genus 3, one can see the above $h(F)$ is just the local contribution of F to $h(f)$.

4 Some lemma's proofs.

4.1 The j-th fundamental cycle of a singular point of branch locus.

Now we claim the following useful result.

Proposition 4.1. *$\tilde{\pi}^*\mathcal{E}_{i+1}$ has a unique factorization ($i \geq 0$)*

$$\tilde{\pi}^*\mathcal{E}_{i+1} = Z_{1,i} + Z_{2,i} + Z_{3,i}, \;\; Z_{j,i} \geq 0,\, j = 1,2,3, \tag{30}$$

where $Z_{j,i}$ is either the fundamental cycle of itself support or zero divisor, and

$$Z_{j,i}Z_{l,i} = 0, \;\; l \neq j.$$

If p_i is totally ramified, then

$$Z_{1,i} \geq Z_{2,i} \geq Z_{3,i}.$$

If p_i is simple ramified, then $Z_{1,i} \geq Z_{2,i}$. Furthermore, $Z_{3,i}$ and $Z_{1,i}$ have no common component.

Moreover,

$$\begin{aligned} -3 =& Z_{1,i}^2 + Z_{2,i}^2 + Z_{3,i}^2, \\ w_i =& p_a(Z_{1,i}) + p_a(Z_{2,i}) + p_a(Z_{3,i}), \end{aligned}$$

where w_i is the local invariants of p_i defined as Sec.?? Specially $1 \leq -Z_{1,i}^2 \leq 3$ and $0 \leq p_a(Z_{1,i}) \leq w_i$.

$Z_{j,i}$ is called as the j-th fundamental cycle of p_i.

Proof. We will check the case that p_i is totally ramified. Take $Z_{1,i}$ as the fundamental cycle on support of $\tilde{\pi}^*\mathcal{E}_{i+1}$. Note that $\tilde{\pi}^*\mathcal{E}_{i+1}$ is a cycle, that is, $\tilde{\pi}^*\mathcal{E}_{i+1}\Gamma \leq 0$ for any irreducible component $\Gamma \leq \tilde{\pi}^*\mathcal{E}_{i+1}$. So $A = \tilde{\pi}^*\mathcal{E}_{i+1} - Z_{1,i}$ is an effective divisor. If $A = 0$, the proof is completed. We assume that $A \neq 0$.

Thus $A^2 \leq -1$ and $AZ_{1,i} \leq 0$. Suppose that $AZ_{1,i} \leq -1$. Since $(\tilde{\pi}^*\mathcal{E}_{i+1})^2 = -3$, $-4 \geq A^2 + Z_{1,i}^2 - 2 \geq -3$, a contradiction. So $AZ_{1,i} = 0$. It implies A is a cycle on itself support and $-2 \leq A^2 \leq -1$.

Let $Z_{2,i}$ be the fundamental cycle on support of A. Then $B = A - Z_{2,i}$ is an effective divisor. If $B = 0$, the discussion is stopped. Suppose $B \neq 0$. By a similar discussion, one has $BZ_{2,i} = 0$, $B^2 = Z_{2,i}^2 = -1$, $A^2 = -2$ and B is a cycle on itself support.

Let $Z_{3,i}$ be the fundamental cycle on support of B. Take $C = B - Z_{3,i}$. Since $B^2 = -1$, $C^2 = 0$, i.e., $C = 0$. It is clear that $Z_{1,i} \geq Z_{2,i} \geq Z_{3,i}$ and $Z_{j,i}Z_{l,i} = 0$ $(l \neq j)$. The uniqueness is gotten by the progress of the proof.

The second part is a straightforward corollary. □

Since F is semistable, any component Γ with multiplicity $\eta_\Gamma > 1$ in $\bar{F}$ must be contracted. Suppose that $Z_{2,i}$ (or $Z_{3,i}$) is non-zero. By Proposition 4.1, $p_a(Z_{2,i}) = 0$ (or $p_a(Z_{3,i}) = 0$), that is, it is an exceptional curve of the first kind (See [Ar66]).

The following lemma is useful to classifying the singular point of branch locus.

Lemma 4.2. *Let Γ be a vertical curve in $\bar{S}$ which is contracted by ρ. Let $\gamma : \bar{S} \to S'$ be a blowing-downs such that $\gamma(\Gamma)$ is a (-1)-curve and $D = \gamma^*\gamma_*\Gamma$ be the totally transform of Γ. Then there is a unique singular point p_i with the above factorization*

$$\tilde{\pi}^*\mathcal{E}_{i+1} = Z_{1,i} + Z_{2,i} + Z_{3,i},$$

such that $D = Z_{j,i}$ for some j. Specially, $\tilde{\pi}^*\mathcal{E}_{i+1} \geq D$ and $\tilde{\pi}^*\mathcal{E}_{i+1}D = -1$.

Furthermore, let $E = \tilde{\pi}(\Gamma)$ in $\bar{P}$. We assume that p_i is not a good cusp. Then Γ occurs in one of the following cases:
(1) E is a vertical (-3)-curve contained in $\bar{D}_2$;
(2) E is a vertical (-2)-curve contained in $\bar{D}_1$;
(3) E is not a component of branch locus and intersects exactly 2 totally ramified (-3)-curves;
(4) p_i is a simple ramified singular point and Γ is a component of $Z_{3,i}$.

Proof. One can choose the minimal $\mathcal{E}_{i+1}$ such that $\tilde{\pi}^*\mathcal{E}_{i+1} \geq D$. Then $\tilde{\pi}^*\mathcal{E}_{i+1}D < 0$ since D has a component of $\tilde{\pi}^*E_{i+1}$. We consider the factorization as in Lemma4.1.

Claim 1. $D = Z_{j,i}$ for some j. Suppose that $D \neq Z_{j,i}$ for any j.

Suppose that $Z_{3,i}D \neq 0$ then $Z_{3,i} \neq 0$. $(Z_{3,i} \pm D)^2 = -2 \pm 2Z_{3,i}D \leq 0$ implies that $Z_{3,i} = D$. It contradicts to the above assumption. Similarly, we can also assert $Z_{2,i}D = 0$. By the construction of fundamental cycle, one has $Z_{1,i} = \gamma^*\gamma_*Z_{1,i}$. Hence $Z_{1,i}D = 0$. Therefore $\tilde{\pi}^*\mathcal{E}_{i+1}D = 0$, a contradiction.

Claim 2. For else $\tilde{\pi}^*\mathcal{E}_{s+1}$, we have $D \neq Z_{j,s}$ $j = 1, 2, 3$.

Suppose that $D = Z_{j,s}$ for some j and $s(\neq i)$. By the minim of $\mathcal{E}_{i+1}$, $\tilde{\pi}^*\mathcal{E}_{s+1} > \tilde{\pi}^*\mathcal{E}_{i+1}$. So $\tilde{\pi}^*\mathcal{E}_{s+1}D = 0$. However, by the assumption, $\tilde{\pi}^*\mathcal{E}_{s+1}D = Z_{i,s}^2 = -1$, a contradiction.

In what follows we always assume Γ doesn't occur in case (4).

Claim 3. E lies in branch locus if and only if then Γ is a (-1)-curve. Thus E satisfies the case (1) or (2).

If Γ is a (-1)-curve, then $\Gamma = D = Z_{j,i}$ $(j \neq 1)$and Γ is a component of $\tilde{\pi}^*E_{i+1}$. So $\tilde{\pi}^*\mathcal{E}_{i+1} \geq Z_{1,i} + Z_{j,i} \geq 2\Gamma$. Namely, $E = \tilde{\pi}(\Gamma)$ lies in branch locus.

If Γ is not a (-1)-curve, then there is another component Γ' in D such that $\Gamma\Gamma' = 0$. Similarly, let $\gamma' : \bar{S} \to S''$ be a blowing-downs such that $\gamma'(\Gamma')$ is a (-1)-curve and $D' = \gamma'^{*}\gamma'_{*}\Gamma'$ be the totally transform of Γ'. It is easy to see $D' < D$ and $\Gamma D' = 1$.

Suppose that E is a component of branch locus. We will induce a contradiction. From Claim 1 and 2, we can find a singular point p_s with a factorization $\tilde{\pi}^*\mathcal{E}_{s+1} = Z_{1,s} + Z_{2,s} + Z_{3,s}$ such that $D' = Z_{l,s}$ for some l.

If $\mathcal{E}_{s+1} > \mathcal{E}_{i+1}$, then $Z_{1,s} \geq D$. By the construction of the fundamental cycle $Z_{1,s}$, we have $Z_{1,s}D = 0$. So $Z_{1,s}\Gamma = 0$. Because $(Z_{2,s} + Z_{3,s})\Gamma \geq D\Gamma \geq 1$, $\tilde{\pi}^*\mathcal{E}_{s+1}\Gamma \geq 1$, a contradiction.

If $\mathcal{E}_{s+1} < \mathcal{E}_{i+1}$ □

Applying Prop. 4.1, one can classify all singular points of branch locus.

Type I_w: p_i is a totally ramified singular point with a factorization $\tilde{\pi}^*\mathcal{E}_{i+1} = Z_{1,i} + Z_{2,i} + Z_{3,i}$ and $w_i = w$. Moreover $Z_{j,i}^2 = -1$ $(j = 1, 2, 3)$ and $p_a(Z_{1,i}) = w$.

Recall that the denotation $E_{i,j}$ is the strict transform of E_i in P_j. Let Γ be a irreducible component of $\tilde{\pi}^*E_{i,k}$. Since $\tilde{\pi}^*\mathcal{E}_{i+1}Z_{j,i} = Z_{j,i}^2 < 0$, Γ is contained in $Z_{j,i}$ $(j = 1, 2, 3)$. Thus $\tilde{\pi}^*\mathcal{E}_{i+1} \geq 3\Gamma$, which implies $\tilde{\pi}^*E_{i+1} = 3\Gamma$ and E_{i+1} is totally ramified. By Lemma 4.2, $\Gamma^2 = -1$, i.e., $E_{i+1,k}^2 = -3$. So there are exactly two infinity close infinitely close singular points of p_i, says p_{i+1} and p_{i+2}, in $E_{i+1,i+1}$.

Suppose p_{i+2} is not the infinitely close singular point of p_{i+1}. $\bar{F} - 3\Gamma$ contains at least 3 connected components. Two of them are $\tilde{\pi}^*(\mathcal{E}_{i+2})$ and $\tilde{\pi}^*(\mathcal{E}_{i+3})$. The third connected component contains a irreducible component of the inverse image of F_0 in $\bar{F}$. After contracting the (-1)-curve Γ, $\bar{F}$ will induce a non-node singular point, a contradiction. Therefore p_{i+2} must be the infinitely close singular point of p_{i+1}.

Noting that the branch locus consists of sections, one can easily find that E_{i+2} is a totally ramified component and E_{i+3} doesn't lie in branch locus. By Lemma4.2, $E_{i+2,k}$ is a (-3)-curve. There is a unique singular point, says p_{i+3}, in E_{i+2} apart from p_{i+2}, and it's exceptional curve E_{i+4} doesn't lie in branch locus.

Thus we get the canonical resolution of singular points of type I_w $(w \geq 0)$ as in Lemma 3.1.

Type II_w: p_i is a totally ramified singular point with a factorization $\tilde{\pi}^*\mathcal{E}_{i+1} = Z_{1,i} + Z_{2,i}$ and $w_i = w$. Moreover $Z_{1,i}^2 = -2$, $Z_{2,i}^2 = -1$ and $p_a(Z_{1,i}) = w - 1$.

By a similar discussion, one can assert that there is a unique infinitely close singular point, says p_{i+1}, in E_{i+1}. Moreover p_{i+1}'s exceptional curve E_{i+2} doesn't lie in branch locus. The canonical resolution in this case can be seen in Lemma 3.1.

Type III_w: p_i is a totally ramified singular point with $\tilde{\pi}^*\mathcal{E}_{i+1} = Z_{1,i}$ and $w_i = w$. Moreover $Z_{1,i}^2 = -3$ and $p_a(Z_{1,i}) = w - 2$.

Suppose E_{i+1} is a component in branch locus. $\tilde{\pi}^*E_{i+1}$ contains a (-1)-curve, says Γ by Lemma 4.2. The construction of the fundamental cycle $Z_{1,i}$ implies $Z_{1,i}\Gamma = 0$. However $Z_{1,i}\Gamma = \tilde{\pi}^*\mathcal{E}_{i+1}\Gamma = \mathcal{E}_{i+1}E_{i+1} = -1$, a contradiction. So E_{i+1} is not a component of branch locus.

Type I_w^*: p_i is a simple ramified singular point with a factorization $\tilde{\pi}^*\mathcal{E}_i = Z_{1,i} + Z_{2,i} + Z_{3,i}$ and $w_i = w$. Moreover $Z_{j,i}^2 = -1$ $(j = 1, 2, 3)$ and $p_a(Z_{1,i}) = w$. In this case, $Z_{1,i}$ and $Z_{3,i}$ have no common component.

In fact it is a singular point of type $(2w+1, 2w+1)$ in view of a local double cover.

Type II_w^*: p_i is a simple ramified singular point with a factorization $\tilde{\pi}^*\mathcal{E}_i = Z_{1,i} + Z_{3,i}$ and $w_i = w$. Moreover $Z_{1,i}^2 = -2$, $Z_{3,i}^2 = -1$ and $p_a(Z_{1,i}) = w - 1$.

It is a singular point of type $2w$ in view of a local double cover.

Now we assert that any curves contracted by ρ must come from the contributions of the above five types of singular points. Namely, we have

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DEPARTMENT OF MATHEMATICS, EAST CHINA NORMAL UNIVERSITY, SHANGHAI 200062, PEOPLE'S REPUBLIC OF CHINA

E-mail address: zjchen@math.ecnu.edu.cn, mathlujun@gmail.com, sltan@math.ecnu.edu.cn