ON THE CLASSIFICATION AND MORDELL-WEIL GROUPS OF FAMILIES OF CURVES WITH TWO SINGULAR FIBERS

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Dedicated to the memory of Professor Gang Xiao

Abstract. A relatively minimal family of curves $f : S \to \mathbb{P}^1$ with 2 or 3 singular fibers is called a Belyi family of curves. In this paper, we study the structure of all Belyi families f of curves of genus $g \geq 2$ with two singular fibers. When S is simple connected, we will write down explicitly the defining equation of the family. We compute all sections of f and its Mordell-Weil group. As an application, we prove that any periodic fiber can be realized as a fiber of a Belyi fibration with two singular fibers.

1. Introduction

A family of curves of genus g over $\mathbb{P}^1$ is a fibration $f : S \to \mathbb{P}^1$, where S is a smooth complex projective surface, and the generic fiber F is a connected smooth curve of genus g. The projection $F \times \mathbb{P}^1 \to \mathbb{P}^1$ is called a trivial fibration. The fibration f is isotrivial if all of the smooth fibers are isomorphic to a fixed curve. We denote by $F_1, F_2, \ldots, F_s$ the singular fibers. We always assume that $g \geq 1$, and f is relatively minimal, i.e., any fiber contains no (-1)-curve. It is well-known that if $f : S \to \mathbb{P}^1$ is non-trivial (resp. non-isotrivial), then $s \geq 2$ (resp. 3, see [Bea81]).

A finite cover $\pi : C \to \mathbb{P}^1$ admits at least 2 branch points, and if C is not isomorphic to $\mathbb{P}^1$, then π admits at least 3 branch points. A curve C admitting a finite cover $\pi : C \to \mathbb{P}^1$ with 2 or 3 branch points is isomorphic to a curve defined over a number field. The famous Belyi theorem says that the converse of the above statement holds true, i.e., any curve defined over a number field is isomorphic to a complex curve C admitting a finite cover over $\mathbb{P}^1$ with 2 or 3 branch points. Due to this theorem, a finite cover $\pi : C \to \mathbb{P}^1$ with 2 or 3 branch points is called a *Belyi cover*. A Belyi cover $\pi : C \to \mathbb{P}^1$ of degree n with two branch points x_1 and x_2 is isomorphic to the cyclic cover $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ defined by $x = z^n$.

Relatively minimal families of curves $f : S \to \mathbb{P}^1$ with 2 or 3 singular fibers have also some interesting arithmetic and geometric properties, e.g., if f is non-isotrivial, then it is isomorphic to a fibration defined over a number field. Similar to the Belyi covers, such a family f is called *a Belyi family* or *a Belyi fibration*.

We consider Belyi fibrations $f : S \to \mathbb{P}^1$ of genus g with two singular fibers F_1 and F_2 over x_1 and x_2, respectively. In [Hir85], U. Schmickler Hirzebruch classifies the elliptic case, there are 5 such families (see also [Tan10] for the equations). In [GLT15], the authors classify completely the genus $g = 2$ case.

The main purpose of this paper is to classify Belyi fibrations $f : S \to \mathbb{P}^1$ of any genus $g \geq 2$ with two singular fibers, and determine their Mordell-Weil groups. Moreover, from the point of view of foliations, such a Belyi fibration is also a Riccati foliation (see [Bru15, Ch. 2, Sec. 1] and Proposition 2.6), this means that the fibers are integral curves of a Riccati differential equation over S. More precisely, we have

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Theorem 1.1. *Let $f : S \to \mathbb{P}^1$ be a Belyi fibration of genus $g \geq 2$ with two singular fibers. Then f is a Riccati fibration (foliation). Namely, S is a ruled surface and the generic fiber F' of the ruling $\varphi : S \to C$ meets transversely with the fibers of f.*

In [GLT15, Theorem 1.1], the authors prove that the Mordell-Weil group of f is finite. The structure of F_1 and F_2 are quite clear, they are dual to each other, i.e., $F_1 = F_2^*$ and $F_2 = F_1^*$. Let $\rho : \overline{S} \to S$ be a blowing-up such that $\overline{F}_i = \rho^* F_i$ is the minimal normal crossing model of F_i (see [LT13, Definition 2.2]) and $\bar{f} = f \circ \rho : \overline{S} \to \mathbb{P}^1$. Then $\overline{F}_i$ is a tree which can be written as

$$\begin{aligned} \overline{F}_1 &= nC_0 + \sum_{i=1}^{b} \Theta_i, \\ \overline{F}_2 &= nC_0^* + \sum_{i=1}^{b} \Theta_i^*, \end{aligned} \tag{1}$$

where Θ_i's (resp. Θ_i^*'s) are disjoint H-J branches

$$\Theta_i = n_{i1}\Gamma_{i1} + \cdots + n_{i\ell_i}\Gamma_{i\ell_i}, \qquad \Theta_i^* = n_{i1}^*\Gamma_{i1}^* + \cdots + n_{i\ell_i^*}^*\Gamma_{i\ell_i^*}^*. \tag{2}$$

and C_0 (resp. C_0^*) is a principal component of genus $q(S)$ (see [LT13, Definition 3.4]), i.e., a nonsingular curve $C_0 \ncong \mathbb{P}^1$ or a smooth rational curve meeting at 3 or more points with the other components. By changing the subscripts, we can assume Θ_i^* is the dual H-J branch of Θ_i for each i (see Sec. 2.1).

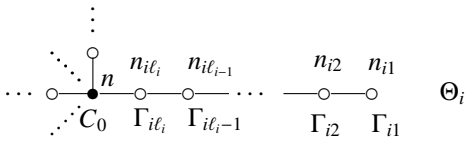


Figure 1

The singular fibers F_1 and F_2 have the same multiplicity m, here the multiplicity of F_1 is the greatest common divisor of the multiplicities of the components in F_1 (see [GLT15]). In case $m > 1$, we consider the pullback fibration $\tilde{f} : \widetilde{S} \to \mathbb{P}^1$ under the cyclic base change $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ of degree m ramified over x_1 and x_2.

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\Pi} & S \\ {\scriptstyle \tilde{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \mathbb{P}^1 & \xrightarrow{\pi} & \mathbb{P}^1 \end{array}$$

Then the two singular fibers of $\tilde{f}$ are not multiple fibers. In fact, $\Pi : \widetilde{S} \to S$ is a cyclic unramified cover of degree m. Namely $S = \widetilde{S}/\mathbb{Z}_m$, and the group $\mathbb{Z}_m$ acts freely on $\widetilde{S}$. Conversely, from $\tilde{f}$ and a certain free group action of $\mathbb{Z}_m$ on $\widetilde{S}$, we can construct f.

We have the following canonical method to construct a Belyi fibration with two singular fibers of multiplicity 1.

Take a smooth projective curve C. Let $D_0 = n_1 p_1 + \cdots + n_b p_b$ be an n-divisible effective divisor on C, where $1 \leq n_i < n$ for any i, i.e., there is a divisor δ_0 on C such that $D_0 \equiv n\delta_0$. We can construct an n-cyclic cover $\pi_0 : F \to C$ ramified over D_0.

Let $P = \mathbb{P}^1 \times C$, $F_i' = \mathbb{P}^1 \times \{p_i\}$, $F_1'' = \{x_1\} \times C$ and $F_2'' = \{x_2\} \times C$. Denote by $pr_1 : \mathbb{P}^1 \times C \to \mathbb{P}^1$ (resp. $pr_2 : \mathbb{P}^1 \times C \to C$) the first (resp. second) projection. Then we have an n-divisible effective divisor

$$D = \sum_{i=1}^{b} n_i F_i' + F_1'' + (n-1)F_2'' \equiv n \cdot (\mathrm{pr}_2^*(\delta_0) + F_1'').$$

Let $\pi : \Sigma \to P$ be the (normalized) n-cyclic cover determined by this n-divisible divisor D. Then Σ has some Hirzebruch-Jung singularities. Let $\sigma : S \to \Sigma$ be the minimal resolution

of the singularities of Σ.

(3)
$$\begin{array}{ccc} S & \xrightarrow{\sigma} & \Sigma \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & \xleftarrow{\mathrm{pr}_1} & P \end{array}$$

Then the induced fibration $f : S \to \mathbb{P}^1$ is a Belyi fibration with two singular fibers over x_1 and x_2. The generic fibers of f are isomorphic to F.

Theorem 1.2. *Let $f : S \to \mathbb{P}^1$ be a Belyi fibration of genus $g \geq 2$ with two singular fibers of multiplicity 1. If the irregularity $q(S) = 0$, then f can be constructed as above, and f admits at most b sections.*

Note that any simply connected singular fiber cannot be a multiple fiber (see [Xia90]). For a Belyi fibration with two singular fibers, the singular fibers are simply connected if and only if S is simple connected by [Xia91]. In this case, we can write down explicitly the equations of the fibrations in Theorem 1.2 by choosing $\alpha_1 = 0$, $\alpha_2 = 1$ and $\alpha_3 = \infty$.

Corollary 1.3. *If $q(S) = 0$, then f is defined explicitly by the equation*

$$z^n = tx^{n_1}(x-1)^{n_2}\prod_{i=4}^{b}(x-\alpha_i)^{n_i},$$

where b is the number of the Hirzebruch-Jung branches in F_1 or F_2. In particular, if $b = 3$, then the family f is unique.

A periodic fiber F_0 is a singular fiber with periodic topological monodromy, equivalently, the semistable reduction of F is a smooth fiber. It is well-known that any periodic singular fiber F_0 can be realized as a central fiber of a local fibration $f : S \to \Delta$ (see [Nie27], [Ker83]). From the canonical construction as above, we can see that F_0 can be the fiber of a global fibration.

Corollary 1.4. *For any periodic fiber F_0, there exits a fibration $f_0 : S \longrightarrow \mathbb{P}^1$ with only two singular fibers F_0 and F_0^*.*

The third author introduces the notion of the Chern numbers $c_1^2(F), c_2(F)$ and χ_F for any singular fiber F in [Tan96]. The second and the third authors conjecture that $c_1^2(F) \geq \chi_F$ in [LT13].

From Corollary 1.4, every periodic fiber F occurs in an isotrivial fibration. Therefore we can easily get the following result from Cornalba-Harris-Xiao's inequality (see [Xia87]).

Corollary 1.5. *If F is a minimal periodic fiber, then we have the following inequalities:*

$$c_1^2(F) \geq \frac{4g-4}{g}\chi_F.$$

Base on the above results, we compute the Mordell-Weil group of f. It is well known that the Mordell-Weil group of a fibration is isomorphic to the quotient group of the Néron-Severi group by the subgroup generated by a section O and all of the components in the fibers (see [Shi99]). Some authors are trying to compute the Mordell-Weil groups of some special fibrations (see [Cox82], [OS91], [SS94], [NS02], [KK05], [Kit05], [GLT13] ...). For example, Miranda and Persson computed the Mordell-Weil groups of all elliptic fibrations with two singular fibers in [MP86]. Our second purpose is to determine the Mordell-Weil group of Belyi fibrations with two singular fibers.

Let λ_i be the topology monodromy of F defined as above. Then we have the following theorem:

Theorem 1.6. *Assume f has a section O intersecting with the b-th H-J branch of F_1. Let $MW(f)$ is the Mordell-Weil group of f and H is a subgroup of $\bigoplus_{i=1}^{b-1}\mathbb{Z}_{\lambda_i}$ generated by $([1],\dots,[1])$. Then $H\cong\mathbb{Z}_n$ and*

$$MW(f)\cong\left(\bigoplus_{i=1}^{b-1}\mathbb{Z}_{\lambda_i}\right)\Big/H.$$

It is easy to see that if f has a section, then $n=$ l.c.m.$(\lambda_1,\dots,\lambda_b)$. From Theorem 1.6 we gets

Corollary 1.7. *The Mordell-Weil group of f is trivial if and only if* $\gcd(\lambda_i,\lambda_j)=1$*, for any $i<j<b$.*

Moreover, for any Belyi fibration $f:S\to\mathbb{P}^1$ with two singular fibers, we can determine all sections of f (See Theorem 3.1).

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2. Preliminaries

2.1. **Hirzebruch-Jung branch and rational canonical divisor.** A curve $\Theta=\Gamma_1+\cdots+\Gamma_r$ in a surface S is called a *Hirzebruch-Jung chain*, if the irreducible components Γ_i's are rational curves satisfying

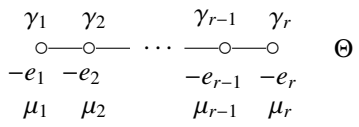
$$\Gamma_i^2=e_i\le -2,\quad \Gamma_i\Gamma_{i+1}=1,\quad \Gamma_i\Gamma_j=0,\quad j\ge i+2.$$

For a Hirzebruch-Jung chain, we can associate it with two sequences of integers (see [BHPV04, Ch III, Sec.5]):

$$0=\gamma_0<\cdots<\gamma_{r+1},\qquad \mu_0>\cdots>\mu_{r+1}=0$$

satisfying $\gamma_1=\mu_r=1$ and

$$\gamma_{i-1}-e_i\gamma_i+\gamma_{i+1}=0,\qquad \mu_{i-1}-e_i\mu_i+\mu_{i+1}=0. \tag{4}$$



By applying induction on i, we can prove

$$\mu_i\gamma_{i+1}-\mu_{i+1}\gamma_i=\gamma_{r+1}=\mu_0.$$

Let $\lambda=\gamma_{r+1}=\mu_0$. From the above equations, we have

$$\mu_i\gamma_r\equiv\gamma_i\pmod\lambda,\quad \mu_1\gamma_i\equiv\mu_i\pmod\lambda,\quad \mu_1\gamma_r\equiv 1\pmod\lambda \tag{5}$$

for each i.

We have

$$\frac{\lambda}{\gamma_r}=e_r-\cfrac{1}{e_{r-1}-\cfrac{1}{\ddots-\cfrac{1}{e_1}}},\qquad \frac{\lambda}{\mu_1}=e_1-\cfrac{1}{e_2-\cfrac{1}{\ddots-\cfrac{1}{e_r}}}.$$

Now the intersection matrix of the curve Θ is

$$M(\Theta) := \begin{pmatrix} -e_1 & 1 & & & \\ 1 & -e_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -e_{r-1} & 1 \\ & & & 1 & -e_r \end{pmatrix}. \tag{6}$$

From a straightforward computation, we have $|\det M(\Theta)| = \lambda$.

A Hirzebruch-Jung chain $\Theta^* = \Gamma_1^* + \cdots + \Gamma_\ell^*$ with $e_i^* = -(\Gamma_i^*)^2$ is said to be dual to $\Theta = \Gamma_1 + \cdots + \Gamma_r$ if

$$\frac{\gamma_{r+1}}{\gamma_{r+1} - \gamma_r} = e_\ell^* - \cfrac{1}{e_{\ell-1}^* - \cfrac{1}{\ddots - \cfrac{1}{e_1^*}}}.$$

The two sequences associated with Θ^* are as follows:

$$0 = \gamma_0^* < \cdots < \gamma_{\ell+1}^*, \qquad \mu_0^* > \cdots > \mu_{\ell+1}^* = 0$$

and $\gamma_1^* = \mu_\ell^* = 1$,

$$\gamma_{i-1}^* - e_i^* \gamma_i^* + \gamma_{i+1}^* = 0, \qquad \mu_{i-1}^* - e_i^* \mu_i^* + \mu_{i+1}^* = 0. \tag{7}$$

It's easy to see that

$$\lambda = \gamma_{\ell+1}^* = \mu_0^* = \gamma_\ell^* + \gamma_r = \mu_1^* + \mu_1.$$

2.2. **Dual fibers.** The second and the third authors introduce the notion of dual model F^* for any singular fiber F of arbitrary genus in [LT13, Definition 2.5]. If the monodromy of F is periodic, then F^* coincides with the notion of Kodaira and others.

Let $\sum_{i=1}^{b} \frac{\sigma_i}{\lambda_i}$ be the valency of the topology monodromy of F_1 (see [AI02, Sec. 1]). Then the valency of the topology monodromy of $F_2 = F_1^*$ is $\sum_{i=1}^{b} \frac{\lambda_i - \sigma_i}{\lambda_i}$. Indeed,

$$\lambda_i = \frac{n}{\gcd(n_{i\ell_i}, n)}, \quad \sigma_i = \frac{n_{i\ell_i}}{\gcd(n_{i\ell_i}, n)}, \quad n_{i\ell_i} + n_{i\ell_i^*}^* = n.$$

If f has a section, then $n = \text{l.c.m.}(\lambda_1, \ldots, \lambda_b)$.

We use the notation in Sec.1. The integers γ_{ij}, μ_{ij} (resp. γ_{ij}^*, μ_{ij}^*) corresponding to the component Γ_{ij} (resp. Γ_{ij}^*) are defined in (4) and (7), and

$$\lambda_i := \gamma_{i,\ell_i+1} = \mu_{i0} = \gamma_{i,\ell_i^*+1}^* = \mu_{i0}^*.$$

It's easy to see that $\gamma_{ij} = \frac{n_{ij}}{n_{i1}}$ and $\gamma_{ij}^* = \frac{n_{ij}^*}{n_{i1}^*}$.

For convenience, we take

$$N_i = \sum_{j=1}^{\ell_i} \frac{\mu_{ij}}{\lambda_i} \Gamma_{ij}, \quad N_i^* = \sum_{j=1}^{\ell_i^*} \frac{\mu_{ij}^*}{\lambda_i^*} \Gamma_{ij}^*. \tag{8}$$

In what follows , we use F or $\overline{F}$ to denote the fibers of $\overline{f}$.

2.3. **The intersection matrix of a singular fiber by removing one component.** We use F or $\overline{F}$ to denote the fibers of $\overline{f}$. In order to get the structure of the Mordell-Weil group, a necessary step is to compute the determinant of any singular fiber F (not necessary relatively minimal) by removing one component. We write

$$F = n_1 C_1 + \cdots + n_d C_d,$$

where C_i is an irreducible component. We denote by M_i the intersection matrix of $F_{\text{red}} - C_i$.

Lemma 2.1. *For any i and j, we have*

$$\frac{n_i^2}{n_j^2} = \frac{\det M_i}{\det M_j}.$$

Proof. Without loss of generality, we assume that $i = 1$ and $j = d > 1$. Let $X = (n_1, \ldots, n_{d-1})^T$. By Zariski's lemma,

$$M_d X = -n_d \begin{pmatrix} C_1 C_d \\ C_2 C_d \\ \vdots \\ C_{d-1} C_d \end{pmatrix}$$

By Cramer's rule, we get

$$n_1 = -n_d \cdot \frac{\det M_d'}{\det M_d}, \tag{9}$$

where

$$M_d' = \begin{pmatrix} C_1 C_d & C_1 C_2 & \ldots & C_1 C_{d-1} \\ C_2 C_d & C_2 C_2 & \ldots & C_2 C_{d-1} \\ \ldots & \ldots & \ldots & \ldots \\ C_{d-1} C_d & C_{d-1} C_2 & \ldots & C_{d-1} C_{d-1} \end{pmatrix}$$

By adding $\frac{n_i}{n_1} \cdot (i\text{-th row})$ to the first row for $i = 2, 3, \ldots, d$, we see that the first row becomes $-\frac{n_d}{n_1}(C_d C_d, C_2 C_d, \ldots, C_{d-1} C_d)$. Hence

$$\det M_d' = -\frac{n_d}{n_1} \det M_1. \tag{10}$$

Combining (9) and (10), we get $\frac{n_1^2}{n_d^2} = \frac{\det M_1}{\det M_d}$. □

Corollary 2.2. *Let $F = \overline{F}_1$ be as in* (1)*, let Γ_{ij} be as in* (2)*, and let M_k be the intersection matrix of $F_{\mathrm{red}} - \Gamma_{k1}$. Then*

$$|\det M_k| = \frac{\prod_{i=1}^b \lambda_i}{\lambda_k^2} \quad \textit{for} \quad k = 1, 2, \ldots, b.$$

In particular, if there is a section O of f intersecting Γ_{k1} for some k, then $\lambda_k = n$ and $|\det M_k| = \frac{\prod_{i=1}^b \lambda_i}{n^2}$.

Proof. Let M_0 be the intersection matrix of $F_{\mathrm{red}} - C_0$. Since all H-J branches Θ_i are disjoint,

$$\det M_0 = \det M(\Theta_1) \ldots \det M(\Theta_b).$$

By (6), we have

$$\det M_0 = \prod_{i=1}^b \lambda_i.$$

Note that the multiplicity of Γ_{k1} (resp. C_0) in $\overline{F}$ is $\frac{n}{\lambda_k}$ (resp. n). From Lemma 2.1, we get $|\det M_k| = \frac{\prod_{i=1}^b \lambda_i}{\lambda_k^2}$. □

By (1), the dual fiber F^* of F has one principle component C_0^* with multiplicity $n^* = n$ in F^* and b H-J branches satisfying $\lambda_1^* = \lambda_1, \ldots, \lambda_b^* = \lambda_b$. Note that a section of f can only meet with one of the components of F with multiplicity one, which must be one of the Γ_{k1}.

Corollary 2.3. *Let F be a periodic fiber and let T (resp. T^*) be the lattice generated by the components of F (resp. F^*) except one component with multiplicity one. Then*

$$|\det T| = |\det T^*|.$$

From [Ngu00, Theorem 1.5], Corollary 2.3 and the fact that $\mathrm{MW}(f)$ is finite, we get

Corollary 2.4. $|\mathrm{MW}(f)| = \frac{\prod_{j=1}^{b}\lambda_j}{n^2}$.

2.4. **Foliations induced by fibrations.** Let $\bar{f}:\overline{S}\to\mathbb{P}^1$ be as in Sec. 1. From [Bru15, Ch. 2, Sec. 3, p. 21], the fibration $\bar{f}:\overline{S}\to\mathbb{P}^1$ can be thought as a relatively minimal foliation $\mathcal{F}$ on $\overline{S}$ whose canonical divisor is

$$K_{\mathcal{F}} = K_{\bar{f}} - \left(\overline{F}_1 - \overline{F}_{1,\mathrm{red}} + \overline{F}_2 - \overline{F}_{2,\mathrm{red}}\right). \tag{11}$$

From [Bru15, Prop. 2.2], for any horizonal irreducible curve Γ, we have

$$K_{\mathcal{F}}\Gamma + \Gamma^2 = tang(\mathcal{F},\Gamma) \geq 0,$$

and $tang(\mathcal{F},\Gamma) = 0$ if and only if Γ is smooth and $\Gamma + F$ is normal-crossing for any fiber F of $\bar{f}$.

Since $g > 1$, $K_{\mathcal{F}}$ is pseudoeffective (see [Bru15, Theorem 7.1]). Then we have the Zariski decomposition

$$K_{\mathcal{F}} = P + N \tag{12}$$

where P is a nef $\mathbb{Q}$-divisor and N is an effective $\mathbb{Q}$-divisor satisfying $PN = 0$. From [Bru15, Theorem 8.1, p. 97] or [Ser92, Theorem 3.4], N is a divisor supported on $\sum_i \Theta_i + \sum_i \Theta_i^*$ and is determined uniquely by the following condition

$$N\Gamma_{ij} = N\Gamma_{ij}^* = \begin{cases} 0, & \text{if } j > 1, \\ -1, & \text{if } j = 1. \end{cases} \tag{13}$$

More precisely, one has

$$N = \sum_{i=1}^{b} N_i + \sum_{i=1}^{b} N_i^*$$

where N_i's and N_i^*'s are as in (8).

From [Bru15, Theorem 9.1] and the isotriviality of $\bar{f}$, the Kodaira dimension of the foliation $\mathcal{F}$ is 1 and $|mP|$ is a base point free linear system for a sufficiently large and divisible integer m, and $|mP|$ induces a new fibration $\varphi:\overline{S}\to C$.

Lemma 2.5. *Let P be as in* (12) *and F be a fiber of $\bar{f}$. Then*

1) $P^2 = 0$,
2) $PK_{\overline{S}} = -\frac{4g-4}{n}$.
3) $P\Gamma_{ij} = P\Gamma_{ij}^* = P\Theta_i = P\Theta_i^* = 0$ *for any i, j.*
4) $PF = K_{\mathcal{F}}F = 2g-2$ *and* $PC_0 = PC_0^* = \frac{2g-2}{n}$.

where n is the multiplicity of C_0 in the singular fiber $\overline{F}_1$ as in (1).

Proof. 1) follows from the fact that $\bar{f}$ is isotrivial. 3) follows from that fact $PN = 0$. In 4), $PF = K_{\mathcal{F}}F = 2g-2$ follows from the definition of $K_{\mathcal{F}}$. Applying it to $F = F_1$ and F_2, we get $PC_0 = PC_0^* = \frac{2g-2}{n}$. Since $K_{\mathcal{F}} = K_{\overline{S}} + F_{1,\mathrm{red}} + F_{2,\mathrm{red}}$,

$$PK_{\overline{S}} = PK_{\mathcal{F}} - PF_{1,\mathrm{red}} - PF_{2,\mathrm{red}} = P^2 + PN - PC_0 - PC_0^* = -\frac{4g-4}{n}.$$

2) holds true. □

Proposition 2.6. *The generic fiber F' of $\varphi:\overline{S}\to C$ is a rational curve, and $K_{\mathcal{F}}F' = 0$. Namely, the foliation $\mathcal{F}$ is a Riccati foliation with respect to the ruling $\varphi:\overline{S}\to C$. Furthermore, C_0 and C_0^* are sections of φ, Θ_i and Θ_i^* lie in the fibers of φ.*

Proof. Note that φ is induced by the linear system $|mP|$. P is numerically equavalent to $\gamma F'$ for some positive rational number. By (2) of Lemma 2.5, $PK_{\overline{S}} < 0$, thus $K_{\overline{S}}F' < 0$. So $K_{\overline{S}}F' = -2$, $F' \cong \mathbb{P}^1$ and $\varphi:\overline{S}\to C$ is the ruling. Since $K_{\mathcal{F}}P = P^2 + NP = 0$, we see that $K_{\mathcal{F}}F' = 0$. Therefore, $\mathcal{F}$ is a Riccti foliation with respect to φ by [Bru15, Ch. 4, Sec. 1]. Namely, $\bar{f}$ is a Riccati fibration.

From (3) of Lemma 2.5, Θ_i and Θ_i^* lie in the fibers of φ. Thus $\Theta_i F' = \Theta_i^* F' = 0$, and

$$0 = K_{\mathcal{F}} F' = (K_{\overline{S}} + F_{1,\mathrm{red}} + F_{2,\mathrm{red}})F' = -2 + C_0 F' + C_0^* F'.$$

Since $C_0 F' \geq 1$ and $C_0^* F' \geq 1$, we get $C_0 F' = C_0^* F' = 1$. Thus C_0 and C_0^* are sections of φ. □

Remark 2.7. *From the discussion above, we have* $F' \sim \frac{n}{2g-2} P$.

3. Horizonal (-1)-curves of f.

The following result implies that there are exactly n horizonal (-1)-curves of f.

Theorem 3.1. *The ruling* $\varphi : \overline{S} \to C$ *has two sections* C_0, C_0^* *and* b *singular fibers* F_i' *for* $i = 1, 2, \ldots, b$. *The support of* F_i' *is* $\Theta_i + \Theta_i^* + E_i$, *where* E_i *is a* (-1)*-curve meeting with* Γ_{i1} *and* Γ_{i1}^*.

Furthermore, if $\bar{f}$ *has a section* D, *then* D *must be one of the* E_i*'s.*

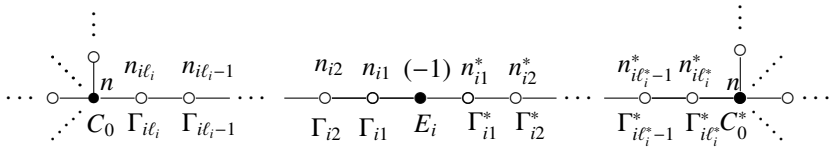


Figure 2

Proof. Let Γ be an irreducible component of a singular fiber F_0' of φ. We assume that $\Gamma \neq \Gamma_{ij}, \Gamma_{ij}^*$ for any i, j. Let N_i, N_i^* and N be as in (8) and (12). Take

$$\overline{N}_i = \sum_{j \geq k_i} \frac{\mu_{ij}}{\lambda_i} \Gamma_{ij}, \quad \overline{N}_i^* = \sum_{j \geq k_i^*} \frac{\mu_{ij}^*}{\lambda_i^*} \Gamma_{ij}^*,$$

and $\overline{N} = \sum_i \overline{N}_i + \sum_i \overline{N}_i^*$, where k_i (resp. k_i^*) is the minimal j such that $\Gamma_{ij}\Gamma > 0$ (resp. $\Gamma_{ij}^*\Gamma > 0$). We take $k_i = \ell_i + 1$ (resp. $k_i^* = \ell_i^* + 1$) if $\Gamma_{ij}\Gamma = 0$ (resp. $\Gamma_{ij}^*\Gamma = 0$) for all j.

From (13), we have

$$(\overline{N}_i + \Gamma)\Gamma_{ij} = \begin{cases} \Gamma\Gamma_{ij}, & \text{if } j > k_i, \\ \Gamma\Gamma_{ij} - \frac{\mu_{i,j-1}}{\lambda_i}, & \text{if } j = k_i. \end{cases} \tag{14}$$

Thus $(\overline{N}_i + \Gamma)\Gamma_{ij} \geq 0$ for $j \geq k_i$. So $(\overline{N}_i + \Gamma)\overline{N}_i \geq 0$. Similarly, we also have $(\overline{N}_i^* + \Gamma)\overline{N}_i^* \geq 0$. Hence

$$(\overline{N} + \Gamma)\overline{N} = \sum_{i=1}^{b} (\overline{N}_i + \Gamma)\overline{N}_i + \sum_{i=1}^{b} (\overline{N}_i^* + \Gamma)\overline{N}_i^* \geq 0.$$

Since $\overline{N} + \Gamma$ lies in some singular fibers of φ, $(\overline{N} + \Gamma)^2 \leq 0$. Thus we have

$$K_{\mathcal{F}}\Gamma + \Gamma^2 = P\Gamma + \Gamma^2 + N\Gamma = \Gamma^2 + \Gamma N = \Gamma^2 + \Gamma\overline{N} \leq -(\overline{N} + \Gamma)\overline{N} \leq 0,$$

where $P\Gamma = 0$ by Remark 2.7. On the other hand, we have $K_{\mathcal{F}}\Gamma + \Gamma^2 \geq 0$. Therefore

$$(\overline{N} + \Gamma)^2 = K_{\mathcal{F}}\Gamma + \Gamma^2 = 0.$$

It implies that $F + \Gamma$ is normal-crossing for any fiber F of $\bar{f}$ and $\overline{N} + \Gamma \sim_{\mathrm{num}} \gamma F'$ for some $\gamma \in \mathbb{Q}$.

From (14), we get

$$0 = \gamma\Gamma_{ij}F' = (\overline{N}_i + \Gamma)\Gamma_{ij} = \begin{cases} \Gamma\Gamma_{ij}, & \text{if } j > k_i, \\ \Gamma\Gamma_{ij} - \frac{\mu_{i,j-1}}{\lambda_i}, & \text{if } j = k_i. \end{cases}$$

Therefore $\Gamma\Gamma_{ij} = 0$ for $j \neq k_i$ and $\Gamma\Gamma_{ik_i} = \frac{\mu_{i,k_i-1}}{\lambda_i}$ (if $k_i \leq \ell_i$). It implies that $k_i = 1$ and $\Gamma\Gamma_{i1} = 1$. Otherwise, $1 \leq \Gamma\Gamma_{ik_i} = \frac{\mu_{i,j+1}}{\lambda_i} < 1$, a contradiction.

As a component of F'_0, Γ must be a (-1)-curve. Note that $F'C_0 = F'C_0^* = 1$, hence F' contains at most two H-J branches. Since one H-J branch in F' plus E form a negative curve (the intersection matrix is negatively definite), which can not be a fiber. So F' contains exactly two H-J branches Θ_i and Θ_j^*. Γ does not meet with C_0 and C_0^* from $F'C_0 = F'C_0^* = 1$. Furthermore,

$$0 = \gamma F'\Gamma = -1 + N_i\Gamma + N_j^*\Gamma = \frac{\mu_{i1}}{\lambda_i} + \frac{\mu_{j1}^*}{\lambda_j^*} - 1.$$

It implies $\lambda_i = \lambda_j^*$ and $\mu_{i1} = \lambda_i - \mu_{j1}^*$. So Θ_j^* is dual to Θ_i.

Note that all H-J branches lie in the fibers of φ. From the above discussion, we have exactly b singular fibers $F'_1, \ldots, F'_b$. By choosing suitable subscripts, we have $F'_i = \lambda_i(N_i + N_i^* + E_i)$ where Θ_i is dual to Θ_i^* and E_i is a (-1)-curve meeting with Γ_{i1} and Γ_{i1}^*.

Let D be a section of $\bar{f}$. There are unique Γ_{i1} and Γ_{j1}^* such that $D\Gamma_{i1} = D\Gamma_{j1}^* = 1$. For other components Γ_{lk} or Γ_{lk}^*, we have $D\Gamma_{lk} = D\Gamma_{lk}^* = 0$.

Since $\mathrm{MW}(f)$ is finite, we can find a basis of $NS(\overline{S}) \otimes \mathbb{Q}$ as follows:

$$P, \quad F, \quad \Gamma_{11}, \ldots, \Gamma_{1\ell_1}, \ldots, \Gamma_{b1}, \ldots, \Gamma_{b\ell_b}, \Gamma_{11}^*, \ldots, \Gamma_{1\ell_1^*}^*, \ldots, \Gamma_{b1}^*, \ldots, \Gamma_{b\ell_b^*}^*$$

where F be a general fiber of $\bar{f}$. By a straightforward computation, we get

$$D \sim_{\mathrm{num}} \left(\frac{1}{2g-2}\right)P + \left(\frac{DP}{2g-2}\right)F - N_i - N_j^*.$$

Thus

$$D^2 = \frac{DP}{g-1} - (N_i + N_j^*)D.$$

On the other hand, we have

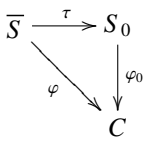
$$0 = K_{\mathcal{F}}D + D^2 = PD + ND + D^2 = PD + D^2 + (N_i + N_j^*)D.$$

So $PD = 0$ and $D^2 = -(N_i + N_j^*)D = -\frac{\mu_{i1}}{n} - \frac{\mu_{j1}^*}{n} < 0$. It shows that D lies in some fiber F' of φ. If F' is smooth then $D = F'$ and $D^2 = 0$, a contradiction. Hence F' is a singular fiber of φ, i.e., $F' = F'_i$ for some i. □

Corollary 3.2. *Let $\overline{F}$ (resp. $\overline{F'}$) be a smooth fiber of $\bar{f}$ (resp. φ). Then $\overline{F} + \overline{F'}$ and $\overline{F} + E_i$ are also normal-crossing.*

4. The proofs of Theorem 1.2 and Corollary 1.3

Consider the ruled surface $\varphi : \overline{S} \to C$ as in Theorem 3.1 and recall the Fig. 1 and 2. Each singular fiber $\overline{F}'_i$ of φ is a chain composed of $\Theta_i + \Theta_i^* + E_i$. We can get a minimal ruled surface $\varphi_0 : S_0 \to C$ by blowing-downs $\tau : \overline{S} \to S_0$ such that all irreducible components of $\overline{F}'_i$ are contracted by τ except $\Gamma_{i\ell_i}$ $(i = 1, \ldots, b)$.



Let $F'_i = \tau(\Gamma_{i\ell_i})$ be the fiber of φ_0 and

$$\Gamma_\infty = \tau(C_0), \quad \Gamma_0 = \tau(C_0^*), \quad e = -C_0^2 = \sum_{i=1}^{b} \frac{\sigma_i}{\lambda_i}.$$

From the above discussion, we see that Γ_∞ and Γ_0 are sections of φ_0 with $\Gamma_\infty^2 = -\Gamma_0^2 = -e$. By a straightforward computation, we have

$$\tau(\overline{F}_1) = n\Gamma_\infty + \sum_{i=1}^{b} \frac{n\sigma_i}{\lambda_i} F_i', \quad \tau(\overline{F}_2) = n\Gamma_0.$$

In what follows, we discuss the case when $C \cong \mathbb{P}^1$ (i.e., $q(S) = 0$). So $\tau(\overline{F}) \in |n\Gamma_0| = |n\Gamma_\infty + neF_0|$ for any fiber $\overline{F}$ of $\bar{f}$, where F_0 is a general fiber of φ_0. Since $h^0(\overline{F}) = 2$, $\bar{f}$ is induced by a subsystem $\Lambda \subseteq |n\Gamma_0|$ which is generated by $\tau(\overline{F}_1)$ and $\tau(\overline{F}_2)$.

We have a general finite cover $\pi_0 : S_0 \dashrightarrow \mathbb{P}^1 \times C$ from the following commutative diagram

$$\begin{array}{ccc} S_0 & \xrightarrow{\varphi_0} & C \\ {\scriptstyle \varphi_\Lambda}\downarrow & \searrow^{\pi_0} & \uparrow{\scriptstyle pr_2} \\ \mathbb{P}^1 & \xleftarrow[pr_1]{} & \mathbb{P}^1 \times C \end{array}$$

where φ_Λ is a rational map defined by Λ.

Let z (resp. x) be the local coordinate of F_0 (resp. Γ_0) nearby Γ_0. The section Γ_0 (resp. Γ_∞) is defined by $z = 0$ (resp. $z = \infty$). Then the elements in Λ can be defined by the equations

$$z^n - t\prod_{i=1}(x - \alpha_i)^{\frac{n\sigma_i}{\lambda_i}} = 0, \quad t \in \mathbb{P}^1 \tag{15}$$

where each α_i is the local coordinate of $p_i = f(F_i') \in C(\cong \Gamma_0)$. In this equation, $t = \infty$ (resp. $t = 0$) corresponds to the element $\tau(\overline{F}_1)$ (resp. $\tau(\overline{F}_2)$) in Λ.

By the equation (15), we can see easily that π_0 is a general cyclic cover. Thus $f : S \to \mathbb{P}^1$ can be induced by a cyclic cover over $\mathbb{P}^1 \times C$ as above. By a straightforward computation, the branch locus is

$$D = \sum_{i=1}^{b} \frac{n\sigma_i}{\lambda_i} L_i' + L_1'' + (n-1)L_2'' \tag{16}$$

where L_i' (resp. L_i'') is the fiber of $pr_2 : \mathbb{P}^1 \times C \to C$ (resp. pr_1) corresponding to p_i (resp. $x_i = f(F_i)$). □

Proof of Corollary 1.4 and 1.5. It is obvious to see that any periodic fiber F_0 as in (1) can be realized as a fiber of a global fibration which is constructed by a cyclic cover of degree n on $C \times \mathbb{P}^1$ with a branch locus as in (3) where $n_i = n - n_{i\ell_i}$ for $i = 1, \ldots, b$.

Let $f_0 : S_0 \to \mathbb{P}^1$ be such a fibration with two singular fibers F_0 and F_0^*. Let $P_1 = f_0(F_0^*)$ and $P_2 \neq f_0(F_0)$. By taking a suitable base change totally ramified at P_1 and P_2, we can get a fibration $f : X \to B$ with m singular fibers. All of them are the copies of F_0. Because f is isotrivial, we have $K_{X/B}^2 = m \cdot c_1^2(F), \deg f_*\omega_{X/B} = m \cdot \chi_F$ (see [Tan96, Theorem A]).

From Cornalba-Harris-Xiao's inequality

$$K_{X/B}^2 \geq \frac{4g-4}{g} \deg f_*\omega_{X/B},$$

we get

$$c_1^2(F) \geq \frac{4g-4}{g}\chi_F.$$

This proof is completed. □

5. Proof of Theorem 1.6

Since $MW(f) = MW(\bar{f})$, it is enough to compute

$$MW(\bar{f}) = NS(\overline{S})/T,$$

where T is a subspace of $NS(\overline{S})$ generated by the section O and all irreducible components in fibers of $\bar{f}$. For any $D \in NS(\overline{S})$, we denote by $[D]$ the class of D in $MW(\bar{f})$.From Theorem 3.1, C_0 is a section of the ruled surface $\varphi : \overline{S} \to C$ intersecting $\Gamma_{1\ell_1}, \ldots, \Gamma_{b\ell_b}$.

Lemma 5.1. *We have*

$$NS(\overline{S}) = \mathbb{Z}F' \oplus \mathbb{Z}C_0 \oplus (\oplus_i \mathbb{Z}E_i) \oplus \left(\oplus_i \oplus_{j<\ell_i} \mathbb{Z}\Gamma_{ij}\right) \oplus \left(\oplus_{i,j} \mathbb{Z}\Gamma^*_{ij}\right).$$

Proof of Theorem 1.6. Now, we assume that $O = E_b$. By Theorem 3.1, the support of F'_i consists of E_i, Γ_{ij}'s and Γ^*_{ij}'s. Let $\lambda_i, \lambda^*_i, \mu_{ij}, \mu^*_{ij}$ be as in §2.1. From Zariski's lemma and the results in §2.1, we have

$$E_i = \frac{1}{\lambda_i}F' - \sum_{j=1}^{\ell_i} \frac{\mu_{ij}}{\lambda_i}\Gamma_{ij} - \sum_{j=1}^{\ell^*_i} \frac{\mu^*_{ij}}{\lambda^*_i}\Gamma^*_{ij}, \quad i = 1, \ldots, b.$$

So

$$F' = \lambda_b O + \sum_{j=1}^{\ell_b} \mu_{bj}\Gamma_{bj} + \sum_{j=1}^{\ell^*_b} \mu^*_{bj}\Gamma^*_{bj} \in T,$$

i.e., $[F'] = [0]$. Hence $MW(\bar{f})$ is generated by $[E_1], \ldots, [E_{b-1}]$. Furthermore, $\lambda_i[E_i] = [F'] = 0$. □

Lemma 5.2. *In $NS(\overline{S})$, we have*

$$C^*_0 = C_0 + m_0F' + \sum_{i=1}^{b} \sigma_i E_i - \sum_{i=1}^{b}\sum_{j=1}^{\ell_i} m_{ij}\Gamma_{ij} - \sum_{i=1}^{b}\sum_{j=1}^{\ell^*_i} m^*_{ij}\Gamma^*_{ij}, \tag{17}$$

where

$$m_0 = \sum_{i=1}^{b} \frac{\sigma_i}{\lambda_i}, \quad m_{ij} = \frac{\sigma_i\mu_{ij} - \gamma_{ij}}{\lambda_i}, \quad m^*_{ij} = \frac{\lambda^*_i\mu^*_{ij} - \sigma^*_i\mu^*_{ij} + \gamma^*_{ij}}{\lambda^*_i}$$

are all integers. In particular, $\sum_{i=1}^{b} \sigma_i[E_i] = [0]$ in $MW(\bar{f})$.

Proof. It is easy to see that $m_0 = -C_0^2$. From (5), m_{ij}, m^*_{ij} are all integers. In particular, $m_{i\ell_i} = 0, m^*_{i\ell^*_i} = 1$,

$$m_{i,j-1} + m_{i,j+1} + m_{ij}\Gamma_{ij}^2 = 0, \quad m^*_{i,j-1} + m^*_{i,j+1} + m^*_{ij}{\Gamma^*_{ij}}^2 = 0$$

and

$$m_{i\ell_i}\Gamma^2_{i\ell_i} + m_{i,\ell_i-1} = 1, \quad m^*_{i\ell^*_i}{\Gamma^*_{i\ell^*_i}}^2 + m^*_{i,\ell^*_i-1} = -1.$$

Denote by D the right hand of (17). By a straightforward computation, we obtain

$$(C^*_0 - D)F' = (C^*_0 - D)C_0 = (C^*_0 - D)\Gamma_{ij} = (C^*_0 - D)\Gamma^*_{ij} = 0.$$

Because of $m_{i1} + m^*_{i1} = \sigma_i$, we have $(C^*_0 - D)E_i = 0$. Lemma 5.1 implies $C^*_0 = D$ in $NS(\overline{S})$. □

Let H be the subgroup of $\bigoplus_{i=1}^{b-1} \mathbb{Z}_{\lambda_i}$ generated by $([1], [1], \ldots, [1])$. Since $\mathrm{lcm}(\lambda_1, \ldots, \lambda_{b-1}) = n$, $H \cong \mathbb{Z}_n$. Now, from Lemma 5.2, we have a group morphism

$$\tau : \left(\bigoplus_{i=1}^{b-1} \mathbb{Z}_{\lambda_i}\right)\Big/H \longrightarrow MW(\bar{f}),$$
$$([n_1], \ldots, [n_{b-1}]) \to \sum_{i=1}^{b-1} \sigma_i n_i [E_i].$$

For any elements $\sum_{i=1}^{b} m_i[E_i]$, there are integers n_i's such that $\sigma_i n_i \equiv m_i \pmod{\lambda_i}$. Noting that $\lambda_i[E_i] = [0]$, we have $\tau([n_1], \ldots, [n_{b-1}]) = \sum_{i=1}^{b} m_i[E_i]$. Since $MW(\bar{f})$ is generated by all $[E_i]$'s, τ is an epimorphism.

From Corollary 2.4,

$$|MW(f)| = \frac{\lambda_1 \ldots \lambda_{b-1}}{n} = \left| \left(\bigoplus_{i=1}^{b-1} \mathbb{Z}_{\lambda_i} \right) \Big/ H \right|.$$

So τ is an isomorphism.

Up to now, the proof is completed. □

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