# On the upper bound of the slope of a trigonal fibration

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**Abstract**

The main purpose of this paper is to compute the upper bound of the slope of trigonal fibrations. An important method of triple cover (see [20]) is applied to the computation of the global invariants of a trigonal fibration.

## 1 Introduction and main results

Let $f : S \to C$ be a fibration of genus $g$ where $S$ is a relatively minimal smooth projective surface over complex number field, $C$ is a smooth projective curve of genus $b$. If the general fibre of $f$ is trigonal, i.e. a triple cover of $\mathbb{P}^1$, $f$ is called a trigonal fibration.

For any relatively minimal fibration $f : S \to C$, we have the following basic relative numerical invariants:

$$\begin{aligned} K_f^2 \stackrel{\text{def}}{=} K_{S/C}^2 &= K_S^2 - 8(g-1)(b-1), \\ \chi_f &= \chi(\mathcal{O}_S) - (g-1)(b-1), \\ e_f &= \chi_{\text{top}}(S) - 4(g-1)(b-1). \end{aligned}$$

Whenever $\chi_f \neq 0$, the slope of the fibration $f$ can be defined as $\lambda_f = K_f^2/\chi_f$.

$\lambda_f = 12$ if and only if $f$ is a Kodaira fibration.

In this paper we will prove the following main result.

**Theorem 1.1.** *Let $\alpha \geq 2$ and $g \geq 4$. Consider the positive definite quadratic form $Q(x,y) = \frac{\alpha(\alpha-1)}{2}(x+y)^2 - (\alpha-1)(g-3)(x+y)y + \frac{g(g-3)}{2}y^2$ where $x, y \in \mathbb{Z}$. Let $q = \min\limits_{y \equiv \alpha \pmod 2} Q(x,y)$. Then*

$$\lambda_f \leq 12 - \frac{24(\alpha-1)\alpha}{\alpha g(g+3\alpha-6) - 2q},$$

*where $\alpha = D_2F$ is defined in Sect 3.*

*Thus Kodaira fibration only occurs when $\alpha = 0$ or $1$.*

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# 2 Preliminaries on the triple covers

In this section we recall some facts about triple covers. The details are referred to [20] and [19].

## 2.1 Triple cover data

Let $P_0$ be a smooth algebra surface over $\mathbb{C}$, and let $\pi : S_0 \to P_0$ be a normal triple cover. A triple cover data is a pair of sections $(s, t, \mathcal{L})$, where $\mathcal{L}$ is an invertible sheaf, $s \in H^0(P_0, \mathcal{L}^2)$ and $0 \neq t \in H^0(P_0, \mathcal{L}^3)$. $S_0$ is the normalization of the surface defined by $z^3 + sz + t = 0$ in the line bundle of $\mathcal{L}$. Any triple cover $\pi$ is determined by some triple cover data $(s, t, \mathcal{L})$.

If $s = 0$, then the triple cover is cyclic and everything is known. So we always assume that $s \neq 0$. Let

$$a = \frac{4s^3}{\gcd(s^3, t^2)}, b = \frac{27t^2}{\gcd(s^3, t^2)}, c = \frac{4s^3 + 27t^2}{\gcd(s^3, t^2)}.$$

Then $a$, $b$ and $c$ are coprime sections of an invertible sheaf such that $a + b = c$.

Conversely, any coprime triples $(a, b, c)$ with $a + b = c$ can determine a triple cover over $P_0$. Assume that we have decompositions (according to the decompositions of their divisors)

$$a = 4a_1a_2^2a_0^3, \quad b = 27b_1b_0^2, \quad c = c_1c_0^2,$$

where $a_1, a_2, b_1, c_1$ are square-free and $\gcd(a_1, a_2) = 1$. Then the data $(s, t)$ determined by $(a, b, c)$ is given as follows:

$$s = a_1a_2^2b_1a_0, \quad t = a_1a_2^2b_1^2b_0.$$

Denote the corresponding divisors by

$$A_i = \mathrm{Div}(a_i), \quad B_i = \mathrm{Div}(b_i), \quad C_i = \mathrm{Div}(c_i).$$

Let $D_1 = B_1 + C_1$, $D_2 = A_1 + A_2$. Then the branch locus of the triple cover $\pi$ is $R = 2D_2 + D_1$. $\pi$ is totally ramified over $D_2 = A_1 + A_2$, hence $D_2$ is called totally ramified branch locus. $D_1$ is called simple ramified branch locus. Let $\mathcal{E}_\pi$ denote the trace-free subsheaf of $\pi_*\mathcal{O}_{S_0}$, then $c_1(\mathcal{E}_\pi) = -\frac{1}{2}R$.

It is known that $P_0$ is smooth if and only if $D_2$ is smooth, $D_2$ and $D_1$ have no common points, and all of the singular points of $D_1$ are good cusps (i.e., locally defined by $y^2 + f(x, y)^3 = 0$, $f(0, 0) = 0$) where $\pi$ is totally ramified.

## 2.2 Canonical resolution

The canonical resolution $\tau : \tilde{S} \to S_0$ of the singularities of $S_0$ is the following commutative diagrams.

$$\begin{array}{ccccccccc}
\tilde{S} = S_k & \xrightarrow{\tau_k} & \cdots\cdots & \longrightarrow & S_2 & \xrightarrow{\tau_2} & S_1 & \xrightarrow{\tau_1} & S_0 \\
\downarrow{\scriptstyle \tilde{\pi}=\pi_k} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0=\pi} \\
\tilde{P} = P_k & \xrightarrow{\sigma_k} & \cdots\cdots & \longrightarrow & P_2 & \xrightarrow{\sigma_2} & P_1 & \xrightarrow{\sigma_1} & P_0
\end{array} \tag{1}$$

(1) $\sigma_{i+1}$ is the blowing-up of $P_i$ at a singular point $p_i$ of the branch locus of $\pi_i$. $S_{i+1}$ is the normalization of $P_{i+1} \times_{P_i} S_i$. $\tilde{\pi} = \pi_k$ has a smooth branch locus. So $\tilde{S} = S_k$ is smooth.

(2) The corresponding data $(a^{(i)}, b^{(i)}, c^{(i)})$ of $\pi_i$ is obtained from

$$(\sigma_i^* a^{(i-1)}, \sigma_i^* b^{(i-1)}, \sigma_i^* c^{(i-1)})$$

by eliminating the common factors.

## 2.3 Computation of invariants

Put

$$d_i = \min\{m_{p_i}(A^{(i)}), m_{p_i}(B^{(i)}), m_{p_i}(C^{(i)})\},$$

where $m_p(D)$ is the multiplicity of a divisor $D$ at $p$. Let

$$m_i = \left[\frac{m_{p_i}(D_1^{(i)})}{2}\right], \tag{2}$$

$$n_i = \begin{cases} m_{p_i}(D_2^{(i)}), & \text{if } d_i \equiv m_{p_i}(D_2^{(i)}) \pmod 3; \\ m_{p_i}(D_2^{(i)}) - 1, & \text{otherwise.} \end{cases} \tag{3}$$

$$w_i = m_i + n_i.$$

Let $E_i$ be the exceptional curve of $\sigma_i$, and $\mathcal{E}_{i+1}$ be the totally transform of $E_i$ in $\tilde{P}$. Let $\sigma = \sigma_1 \cdots \sigma_k : \tilde{P} \to P_0$.

By a computation, we get the branch locus of $\tilde{\pi}$,

$$\begin{aligned} \tilde{D}_1 &= \sigma^*(D_1) - 2\sum_{i=0}^{k-1} m_i \mathcal{E}_{i+1}, \\ \tilde{D}_2 &= \sigma^*(D_2) - \sum_{i=0}^{k-1} n_i \mathcal{E}_{i+1}. \end{aligned} \tag{4}$$

As $\tilde{D}_1 \tilde{D}_2 = 0$, by formula (4), we have

$$D_1 D_2 = 2\sum_{i=0}^{k-1} m_i n_i. \tag{5}$$

Now we have the following formulae for triple covers:

$$\begin{aligned} \chi(\mathcal{O}_{\tilde{S}}) = 3\chi(\mathcal{O}_{P_0}) &+ \frac{1}{8}D_1^2 + \frac{1}{4}D_1 K_{P_0} + \frac{5}{18}D_2^2 + \frac{1}{2}D_2 K_{P_0} \\ &- \sum_{i=0}^{k-1} \frac{m_i(m_i-1)}{2} - \sum_{i=0}^{k-1} \frac{n_i(5n_i-9)}{18}, \end{aligned} \tag{6}$$

$$\begin{aligned} K_{\tilde{S}}^2 = 3K_{P_0}^2 &+ \frac{1}{2}D_1^2 + 2D_1 K_{P_0} + \frac{4}{3}D_2^2 + 4D_2 K_{P_0} \\ &- \sum_{i=0}^{k-1} 2(m_i-1)^2 - \sum_{i=0}^{k-1} \frac{4n_i(n_i-3)}{3} - k. \end{aligned} \tag{7}$$

## 2.4 Singular points of branch locus of a triple cover

Let $\pi : S_0 \to P_0$ be a triple cover and $\tilde{\pi} : \tilde{S} \to \tilde{P}$ be the canonical resolution.

$$\begin{array}{ccccccccc}
\tilde{S} = S_k & \xrightarrow{\tau_k} & \cdots\cdots & \longrightarrow & S_2 & \xrightarrow{\tau_2} & S_1 & \xrightarrow{\tau_1} & S_0 \\
\downarrow{\scriptstyle \tilde{\pi}=\pi_k} & & & & \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_0=\pi} \\
\tilde{P} = P_k & \xrightarrow{\sigma_k} & \cdots\cdots & \longrightarrow & P_2 & \xrightarrow{\sigma_2} & P_1 & \xrightarrow{\sigma_1} & P_0
\end{array}$$

Let $p \in P_0$ be a totally ramified singular point of the branch locus of $\pi$. Let $Z$ be the fundamental cycle supported in $\mathrm{supp}(\tilde{\pi}^*\mathcal{E}_1)$.

Let $A = \tilde{\pi}^*\mathcal{E}_1 - Z$. Since $WC_i \le 0, \forall C_i \le W$, we have $A \ge 0$. If $A \ne 0$, then $-3 = W^2 = Z_W^2 + A^2 + 2AZ_W \le -2 + 2AZ_W$.s So $AZ_W = 0$ and $A^2 \ge -2$.

Let $Z_A$ be the fundamental cycle supported in $A$ and $B = A - Z_A$. Since $AC_i = WC_i \le 0$, $\forall C_i \le A$, $B \ge 0$. If $B = 0$, then $W_p = Z_W + Z_A$. If $B \ne 0$, then $-2 \le A^2 = Z_A^2 + B^2 + 2Z_AB \le -2 + 2Z_AB$, so $Z_AB = 0$ and $Z_A^2 = B^2 = -1$.

Since $BC_i = AC_i \le 0$, $\forall C_i \le B$, we have $D = B - Z_B \ge 0$. Assume that $D \ne 0$, $-1 = B^2 = D^2 + Z_B^2 + 2DZ_B \le -2$, a contradiction. So $B = Z_B$.

Then we have an unique decomposition of $W_p$ as follows.

$$\begin{aligned}
\tilde{\pi}^*\mathcal{E}_1 &= Z_1 + Z_2 + Z_3, \\
Z_1 > Z_2 &\ge Z_3 \ge 0, \\
Z_iZ_j &= 0, \quad i \ne j.
\end{aligned} \tag{8}$$

If $Z_i \ne 0$, then $\mathrm{supp}(Z_i)$ is a connected negative definite curve and $Z_i$ is a fundamental cycle supported in $\mathrm{supp}(Z_i)$.

For simplicity, $Z_i$ is called $i$-th fundamental cycle of $p$.

For a singularity of a double cover, we also have a similar decomposition as above: $\tilde{\pi}^*\mathcal{E}_1 = Z_1 + Z_2$. Let $p \in X$ be a simply ramified singular point of the branch locus of $\pi$, that is, $\pi^{-1}(p)$ consists of a singularity of a local double cover and a smooth point. Let $Z_3$ be the fundamental cycle supported in the exceptional curve of the smooth point in $\pi^*(p)$. Then we also have $\tilde{\pi}^*\mathcal{E}_1 = Z_1 + Z_2 + Z_3$.

For any singular point $p_i$ of branch locus of $\pi_i$ in the resolution, we have a similar decomposition:

$$\tilde{\pi}^*\mathcal{E}_{i+1} = Z_1^{(i)} + Z_2^{(i)} + Z_3^{(i)}, \tag{9}$$

Since $-3 = (\tilde{\pi}^*\mathcal{E}_{i+1})^2 = (Z_1^{(i)})^2 + (Z_2^{(i)})^2 + (Z_3^{(i)})^2$, we have $-3 \le (Z_1^{(i)})^2 \le -1$.

Since $Z_1^{(i)} \ge Z_2^{(i)}$, we have $(Z_1^{(i)})^2 = \tilde{\pi}^*\mathcal{E}_{i+1}Z_1^{(i)} \le \tilde{\pi}^*\mathcal{E}_{i+1}Z_2^{(i)} = (Z_2^{(i)})^2$. Similarly, we have $(Z_2^{(i)})^2 \le (Z_3^{(i)})^2$ if $p_i$ is totally ramified. If $p_i$ is simply ramified, $(Z_3^{(i)})^2 = -1$.

Hence each singular point $p_i$ occurs in one of the following cases.

**Case** 1. $(Z_1^{(i)})^2 = -1$.

Then we have $(Z_2^{(i)})^2 = (Z_3^{(i)})^2 = -1$

If $p_i$ is totally ramified, then $Z_2^{(i)} \ge Z_3^{(i)}$. Let $C_1 \le Z_3^{(i)}$ such that $Z_3^{(i)}C_1 < 0$. As $Z_3^{(i)}$ is the fundamental cycle and $(Z_3^{(i)})^2 = -1$, $Z_3^{(i)}C_1 = -1$. Since

$Z_j^{(i)} \leq Z_k^{(i)}$ and $Z_j^{(i)} Z_k^{(i)} = 0$ $(j > k)$, $\mathcal{E}_{i+1} C_1 = -3$ and $\mathcal{E}_{i+1} \geq 3C_1$. It implies that $\tilde{\pi}^* E_{i+1} = 3C_1$, that is, $E_{i+1}$ lies in totally ramified branch locus.

If $p_i$ is simply ramified, then, by a similar discussion, $E_{i+1}$ lies in simply ramified branch locus. Let $\tilde{\pi}^* E_{i+1} = 2C_1 + C_2$. We have $Z_2^{(i)} C_1 = -1$ and $Z_3^{(i)} C_2 = -1$.

**Case** 2. $(Z_1^{(i)})^2 = -2$.

If $p_i$ is totally ramified, then $(Z_2^{(i)})^2 = -1$ and $Z_3^{(i)} = 0$. Let $C_1 \leq Z_2^{(i)}$ such that $Z_2^{(i)} C_1 < 0$. As $Z_2^{(i)}$ is the fundamental cycle and $(Z_2^{(i)})^2 = -1$, $Z_2^{(i)} C_1 = -1$. Since $Z_2^{(i)} \leq Z_1^{(i)}$ and $Z_1^{(i)} Z_2^{(i)} = 0$, $\mathcal{E}_{i+1} C_1 < 0$ and $\mathcal{E}_{i+1} \geq 2C_1$. It implies that $E_{i+1}$ lies in branch locus. If $E_{i+1}$ is totally (resp. simply) ramified, then the coefficient of $C_1$ in $Z_1^{(i)}$ is just 2 (resp. 1).

If $p_i$ is simplify, then $(Z_3^{(i)})^2 = -1$ and $Z_2^{(i)} = 0$. By the definition of $Z_3^{(i)}$, $Z_1^{(i)}$ and $Z_3^{(i)}$ have no common component.

**Case** 3. $(Z_1^{(i)})^2 = -3$.

In this case, we have $Z_2^{(i)} = Z_3^{(i)} = 0$. Then $p_i$ must be totally ramified since $Z_3^{(i)} \neq 0$ holds for any simply ramified point.

# 3 Normalized model

we have the following results.

**Lemma 3.1.** *If $f : S \to C$ is a trigonal fibration , then after some base change, we have the following commutative diagram:*

$$\begin{array}{ccccc} \tilde{P} & \xleftarrow{\tilde{\pi}} & \tilde{S} & & \\ \downarrow{\scriptstyle \tilde{\sigma}} & & \downarrow{\scriptstyle \tilde{\tau}} & & \\ P_0 & \xleftarrow{\Phi} & S' & \longrightarrow & S \\ & \searrow^{\varphi_0} & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & C' & \longrightarrow & C \end{array} \tag{10}$$

*where $f' : S' \to C'$ is the base change of $f$, $\tilde{\sigma}$, $\tilde{\tau}$ are birational morphisms, $\varphi_0 : P_0 \to C'$ is a minimal ruled surface, $\tilde{\pi} : \tilde{S} \to \tilde{P}$ is a smooth triple cover.*

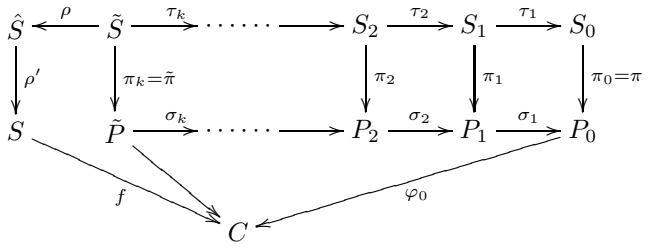
**Lemma 3.2** ([18], Corollary 4.3 and Corollary 4.4)**.** *Let $\tilde{f}$ be the semistable reduction of $f$ with $\lambda_f > 8$.*

(1) *If $f$ is not semistable, then $\lambda_{\tilde{f}} > \lambda_f$.*

(2) *If the slope of $f$ is maximal, then $f$ is semistable.*

In what follows, we can always assume that $f$ is semistable. We have the

following commutative diagram




(11)

where $\pi : S_0 \to P_0$ is a triple cover and $\varphi_0 : P_0 \to C$ is a relatively minimal ruled surface. $\tilde{\pi} = \pi_k : \tilde{S} \to \tilde{P}$ is the canonical resolution of $\pi$. $\rho : \tilde{S} \to \hat{S}$ contracts all $(-1)$-curves in the exceptional curves of the singularities of $\pi$ in the canonical resolution. $\rho'$ contracts all $(-1)$-curves in fibers of $f\rho' : \hat{S} \to C$. Then we get a relatively minimal fibration $f : S \to C$ of genus $g$.

Let $F_0$ be a fiber of $\varphi_0 : P_0 \to C$ , and let $C_0$ be a section such that the self-intersection number $C_0^2 = -e$ is minimal.

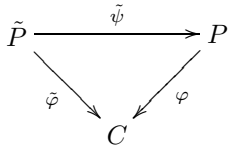
Recall branch locus $R = D_1 + 2D_2$ of $\pi$. Since the genus of a general fiber is equal to $g$, $RF = D_1F + 2D_2F = 2g + 4$. Let $D_2 \sim \alpha C_0 + \beta F$, $D_1 \sim (2g + 4 - 2\alpha)C_0 + 2\gamma F$.

We can choose a better minimal model of the ruled surface $\tilde{P} \to C$.

**Lemma 3.3.** *After some base changes we can assume that $R$ has no vertical fibers, and that each component of $D_1$ or $D_2$ is a section of $\varphi_0 : P \to C$.*

In what follows, we always assume that the branch locus $R$ satisfies Lemma 3.3.

**Lemma 3.4.** *$\tilde{P}$ can be contracted to a relatively minimal model $P$ with a ruling $\varphi : P \to C$ satisfying the following conditions.*




*(1) Let $\tilde{R} = \tilde{D}_1 + 2\tilde{D}_2$ be the branch locus of $\tilde{\pi}$, where $\tilde{D}_1$ (resp. $\tilde{D}_2$) is the simply (resp. totally) ramified components. Let $R = D_1 + 2D_2$ be the image of $\tilde{R}$ in $P$. Then $\tilde{\psi} : \tilde{P} \to P$ is the canonical resolution of $R$.*

*(2) Let $R_h$ (resp. $D_{1,h}$, $D_{2,h}$) be the horizontal part of $R$ (resp. $D_1$, $D_2$). Let $\alpha = D_2F_0$.*

*If $\alpha \geq 2$, then the orders of the singular points of $D_{2,h}$ are less or equal to $\alpha/2$.*

*If $\alpha \leq 1$, then the orders of the singular points of $R_h$ are less or equal to $g + 2$.*

*(3) Let $p$ be an intersection of $D_2$ and $F_0$. Furthermore, if $\alpha = 2$ and each component of horizonal part of $R$ is a section, then the orders of $D_{1,h}$ at $p$ is less or equal to $g$.*

*Such a geometrically ruled surface $\varphi : P \to C$ with the branch locus $R$ will be called normalized.*

*Proof.* A similar proof of (1) and (2) can be referred to [4] Lemma 5.2. We only need to consider (3). In this case, $\alpha = 2$ and $D_{1,h}F_0 = 2g$.

Let $R$ satisfy (1) and (2). By the hypothesis, $D_2$ consists of two disjoint section, denoted by $\Gamma_1 + \Gamma_2$. Let $p_i$ be the intersection of $\Gamma_i$ and $F_0$. By (2), we have $p_1 \neq p_2$.

If the order of $D_{1,h}$ at $p_1$ is more than $g$, then the order of $D_{1,h}$ at $p_2$ is less than $g$. We blow up $p_1$, then $F_0$ becomes a $(-1)$-curve. After contract this $(-1)$-curve, we get a new fiber, denoted by $F_1$. Similarly, we do the same procedure until the singular points of $R_h$ in the fiber satisfy (3). □

# 4 Relative ramification index

Let $D$ be a effective divisor in the ruled surface $\varphi_0 : P_0 \to C$. Then the relative ramification index of $D$ is defined as

$$r_D = D(D + K_{P_0/C}).$$

Suppose that $D$ be a horizontal reduced effective divisor (i.e. $D$ does not contain any fibres of $\varphi_0$). If after $k$ blow-ups $\sigma_i$ the strict transform $\tilde{D}$ of $D$ becomes smooth (it may be composed of several disjoint nonsingular curves), then

$$r_D = 2\sum_{p\in D}\delta_p(D) + \{\text{ramification index of this finite morphism}\,\tilde{D} \to C\}, \quad (12)$$

where $\sum\limits_{p\in D}$ runs over all singular points of $D$, and $\delta_p(D)$ is the geometric genus of $p \in D$. The details can be referred to [23].

Let $D$ and $D'$ be effective divisors in the ruled surface $\varphi_0 : P_0 \to C$, and $F_0$ be a fibre. Let $D \sim aC_0 + bF_0$, $D' \sim a'C_0 + b'F_0$. The following equalities follows from a straightforward computation.

$$(K_{P_0/C}D)(F_0D') + (K_{P_0/C}D')(F_0D) + 2DD' = 0. \quad (13)$$

In particular, if $D = D'$, we have

$$(F_0D)r_D - D^2(F_0D - 1) = 0, \quad (14)$$

$$r_D + (K_{P_0/C}D)(F_0D - 1) = 0. \quad (15)$$

Take $D = D_1$, $D' = D_2$, we have

$$D_1D_2 = \frac{\alpha}{4g + 6 - 4\alpha}r_{D_1} + \frac{g + 2 - \alpha}{\alpha - 1}r_{D_2}, \qquad (\text{if } \alpha_2 \neq 1). \quad (16)$$

From equality (4), we have

Let $\tilde{\pi} : \tilde{S} \to \tilde{P}$ be the canonical resolution of $\pi$ (see (1)). Let $\tilde{R} = \tilde{D}_1 + 2\tilde{D}_2$ be the branch locus of $\tilde{\pi}$. Then

$$r_{D_1} = \sum_{i=0}^{k-1} 2m_i(2m_i - 1) + r_{\tilde{D}_1}, \quad (17)$$

$$r_{D_2} = \sum_{i=0}^{k-1}(n_i - 1)n_i + r_{\tilde{D}_2}, \quad (18)$$

where $m_i, n_i$ are invariants defined as in Sect. 2.3.

In particular, if each irreducible components of $\tilde{R} = \tilde{D}_1 + 2\tilde{D}_2$ is a sections of ruled surface $\tilde{P} \to C$, then $r_{\tilde{D}_i} = -2t_i$ where $t_i$ is the number of all exceptional components of the blowing-ups $\sigma : \tilde{P} \to P_0$ which lie in $\tilde{D}_i$.

Then we have

**Corollary 4.1.** *Let $\varepsilon$ be the number of $(-1)$-curves contracted in all fibers of $\tilde{S} \to C$, i.e., $\varepsilon = \chi_{\mathrm{top}}(\tilde{S}) - \chi_{\mathrm{top}}(S)$. Then we have*

$$3(g+1)K_f^2 = \left(\frac{3g-1}{2} - \frac{g-3}{4g+6-4\alpha}\right) r_{D_1} + 4(g-1)r_{D_2} - 16\sum_{i=0}^{k-1} m_i n_i$$
$$- (g+1)\sum_{i=0}^{k}(6(m_i-1)^2 + 4n_i(n_i-3) + 3) + 3(g+1)\varepsilon \qquad (19)$$

$$36(g+1)\chi_f = \left(\frac{9g+5}{2} - \frac{g-3}{4g+6-4\alpha}\right) r_{D_1} + 2(5g+1)r_{D_2} - 16\sum_{i=0}^{k-1} m_i n_i$$
$$- 2(g+1)\sum_{i=0}^{k}(9m_i(m_i-1) + n_i(5n_i-9)) \qquad (20)$$

$$e_f = r_{D_1} + 2r_{D_2} - \sum_{i=0}^{k}(2m_i(2m_i-1) + 2n_i(n_i-1) - 3) - \varepsilon, \qquad (21)$$

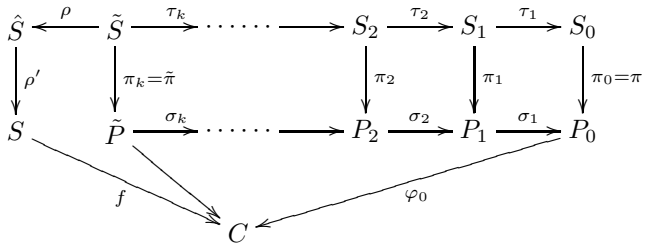
*Proof.* It follows from equalities (6), (7), (14), (15) and (16). □

# 5 $(-1)$-curves in fibers

## 5.1 Notations

Throughout this paper we always assume that $f : S \to C$ is a semistable fibration induced by a triple cover on a normalized ruled surface which satisfies Lemma 3.4 and Lemma 3.3.

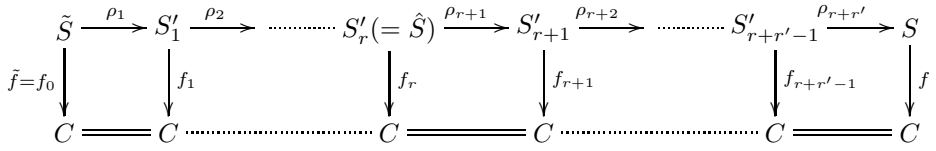
Recall the commutative diagram (11).




where $\tilde{\pi}$ is the canonical resolution of $\pi$. Let $E_i$ be the exceptional curve of $\sigma_i$, and $\mathcal{E}_{i+1}$ be the totally transform of $E_i$ in $\tilde{P}$. Let $\sigma = \sigma_1 \cdots \sigma_k : \tilde{P} \to P_0$.

$\rho : \tilde{S} \to \hat{S}$ contracts all $(-1)$-curves in the exceptional curves of the singularities of $\pi$ in the canonical resolution. $\rho'$ contracts all $(-1)$-curves in fibers of $f\rho' : \hat{S} \to C$. Then we get a relatively minimal fibration $f : S \to C$ of genus $g$.

$\rho$ and $\rho'$ consist of some blow-downs, that is, $\rho = \rho_r\rho_{r-1}\cdots\rho_1$ and $\rho' = \rho_{r+r'}\cdots\rho_{r+1}$.



where $\rho_i : S'_{i-1} \to S'_i$ be the contraction of $\Gamma_i$ and the strict transform of $\Gamma_i$ in $\tilde{S}$ is denoted by $\tilde{\Gamma}_i$; where the fibrations $f_k : S'_k \to C$ are induced by $f$ and $\rho_i$

## 5.2 Extra contraction

An irreducible component of $\tilde{F}$ is called contractible if it is just some $\tilde{\Gamma}_i$. Furthermore, if $i > r$, then $\tilde{\Gamma}_i$ is called extra contractible. Put $\varepsilon_P = r$ and $\varepsilon_F = r'$.

Let $F$ be a fiber of $f$ and $F_k$ be the inverse image fiber of $F$ in $S'_k \to C$, and $\tilde{F} = \rho^*\rho'^*F$ be the fiber of $\tilde{S} \to C$. Let $F_0 = \sigma\tilde{\pi}(\tilde{F})$ be the image fiber of $\tilde{F}$ in $\varphi_0 : P_0 \to C$. Let $\tilde{F}_0$ be the strict transform of $F_0$ under $\sigma$. The multiplicity of an irreducible component $\Gamma$ in a fiber is always denoted by $n_\Gamma$.

As $f$ is semistable, all singular points of fiber $F_k$ in $S'_k$ are nodes ($k = 1, \cdots, r + r'$).

We assert that $F_r$ has no multiple component. In fact, if there is a multiple component in $F_r$, denoted by $\Gamma$, then $\Gamma$ must be the strict transform of some $\Gamma_i$ ($i > r$). It is easy to see that any component of $\rho^*\Gamma$ is also a multiple component of of $\tilde{F}$. In particular, we can find an irreducible component $C_1$ of $\rho^*\Gamma$ such that $\tilde{\pi}(C_1) = \tilde{F}_0$, by the definition of extra contraction. As $F_0$ is not the component of branch locus and $n_{\tilde{F}_0} = 1$ in the fiber $\sigma^*F_0$ of $\tilde{P} \to C$, $n_{C_1} = 1$ in $\tilde{F}$, a contradiction.

Moreover, it is easy to see that all $(-1)$-curves and $(-2)$-curves in $F_r$ consist of some chains. Specially, $\Gamma_{r+1}$ pass through only one node in $F_r$ if $\Gamma_{r+1}$ is a $(-1)$-curve. Then we can assert that there is at most one totally ramified non-good-cusp singular point in $F_0$. Furthermore, if $\tilde{\pi}^*\tilde{F}_0$ is irreducible, i.e., $\tilde{\pi}^*\tilde{F}_0 = \tilde{\Gamma}_{r+1}$, there is at most one singular point of branch locus in $F_0$. In fact, if the above assertion is false, then one can find easily that $\Gamma_{r+1}$ pass through at least two nodes in $F_r$, a contradiction.

The facts will help us to prove the following lemma.

**Lemma 5.1.** *If $\alpha = D_2F_0 \geq 2$, then $\varepsilon_F = 0$, i.e., there is no extra contractible curve.*

*Proof.* Assume that extra contraction occurs. We will get a contradiction.

Let $r_0$ be the number of good cusps in $F_0$. Let $r_1$ (resp. $r_2$) be the number of simply (resp. totally) ramified smooth points of branch locus in $F_0$. As $\tilde{\Gamma}_{r+1}$ is rational, $2r_0 + r_1 + 2r_2 \leq 4$, by Hurwitz formula.

Note that all components of of $D_2$ are sections. If $r_2 \leq 1$, then we can find one point $p_0$ in $F_0$, s.t., $\mathrm{mult}_{p_0}(D_2) = \alpha - r_2$, by the above facts. As $P_0$ is normalized model, $\mathrm{mult}_{p_0}(D_2) \leq \frac{\alpha}{2}$. So $\alpha = 2$ and $r_2 = 1$. In this case, $\tilde{\pi}^*\tilde{F}_0$ is irreducible. Moreover, we have $2r_0 + r_1 \leq 2$ and $\mathrm{mult}_{p_0}(D_1) = 2g - 2r_0 - r_1$,

also by the above facts. As $P_0$ is normalized model, $\mathrm{mult}_{p_0}(D_1) \leq g$. Thus $g \leq 2$, a contradiction.

If $r_2 = 2$, then $r_0 = r_1 = 0$ and $\tilde{\pi}^*\tilde{F}_0$ is irreducible. All components of of $D_1$ pass through one point $p_0$ in $F_0$, i.e., $\mathrm{mult}_{p_0}(D_1) = 2g$, by the above facts. Let $E_1$ be the exceptional curve of $p_0$ and $\mathcal{E}_1$ be total transform of $E_1$ in $\tilde{P}$. The strict transform of $E_1$ in $\tilde{P}$ is denoted by $\tilde{E}_1$.

Note that $\tilde{\Gamma}_{r+1} = \tilde{\pi}^*\tilde{F}_0$ is an irreducible rational curve and $r_2 = 2$, $\tilde{E}_1$ doesn't lie in branch locus. Hence each irreducible component of $\tilde{\pi}^*E_1$ normal crosses $\tilde{\Gamma}_{r+1}$. Then one can find that $\Gamma_{r+1}$ pass through at least two nodes in $F_r$, a contradiction.

Up to now, we proved this result. □

## 5.3 $(-1)$-curves in the exceptional set

For simplicity, the strict transform of exceptional component $E_i$ in $P_j$ is also denoted by $E_i$ without a confusion. $n_{E_i}$ is the multiplicity of $E_i$ in the fiber $\sigma^*F_0$. We always assume that $\alpha \geq 2$. So extra contraction doesn't occur in this case.

**Lemma 5.2.** *(1)Assume that $E_i$ is a totally ramified component of branch locus, i.e., $\tilde{\pi}^*(E_i) = 3\Gamma$, then $\Gamma$ is just a $(-1)$-curve, that is, $E_i$ is a $(-3)$-curve in $\tilde{P}$.*

*(2)Assume that $E_i$ is a simply ramified component of branch locus, i.e., $\tilde{\pi}^*E_i = 2\Gamma + \Gamma'$, then $\Gamma$ is a $(-1)$-curve. that is, $E_i$ is a $(-2)$-curve in $\tilde{P}$.*

*Proof.* We only prove (1). (2) can be proved similarly. Suppose $\Gamma$ be not a $(-1)$-curve. We will get a contradiction. Without a loss of generality, we can assume the subscript $i$ of $E_i$ is maximal, that is, if there is a totally ramified exceptional component $E_k$ such that $\mathrm{supp}(\tilde{\pi}^*(E_k))$ is not a $(-1)$-curve, then $k \leq i$. Then we can find a component $C_1$ in $\tilde{F}$ such that $C_1\Gamma > 0$ and $\rho$ contract $C_1$. Then $n_{C_1} \geq n_\Gamma$, by the above discussion. Let $E_l = \tilde{\pi}(C_1)$. As branch locus of $\tilde{\pi}$ is smooth, $E_l$ doesn't lie in branch locus. Note that $\tilde{F} = (\sigma\tilde{\pi})^*F_0$, we have $n_{C_1} = n_{E_l}$ and $n_\Gamma = 3n_{E_i}$. So $n_{E_l} \geq 3n_{E_i}$. We assert that it is impossible.

If $l < i$ then $n_{E_l} \leq n_{E_i}$, a contradiction. So $i > l$, that is, we can find a singular point $p_i \in E_i$ (in ruled surface $P_i$) such that the above $E_l$ occurs in the totally transform of $p_i$ under $\sigma$. Blowing-up $p_i$, we get an exceptional component $E_{i+1}$ in $P_{i+1}$. If there is no else $E_k$ pass through $p_i$ $(k < i)$, then $n_{E_{i+1}} = n_{E_i}$; else, we have $n_{E_{i+1}} = n_{E_k} + n_{E_i} \leq 2n_{E_i}$ since $n_{E_k} \leq n_{E_i}$.

As $n_{E_{i+1}} \leq 2n_{E_i}$, and the totally transform of $E_{i+1}$ contains $E_l$, $E_{i+1}$ must be a component of branch locus. As the intersection number of $D_1$ and $D_2$ at any point is even, $E_{i+1}$ must be totally ramified component. Let $p_{i+1} = E_i \cap E_{i+1}$ in $P_{i+1}$.

Note that $p_i$ is a singular point of branch locus and all components of branch locus are sections. Hence there is another singular point of branch locus in $E_{i+1}$ (in $P_{i+1}$), denoted by $p_{i+2}$. By the hypothesis for the the subscript $i$, after blowing up $p_{i+1}$ and $p_{i+2}$, $E_{i+1}$ becomes an isolated $(-3)$-curve. Let $E_{i+2}$ be the exceptional component of $\sigma_{i+2}$ at $p_{i+1}$. Namely, $E_{i+2}$ doesn't lie in branch locus. Thus $E_l = E_{i+2}$ since $E_lE_i > 0$ in $P_{i+2}$ and $E_l$ occurs in the totally transform of $p_i$ under $\sigma$.

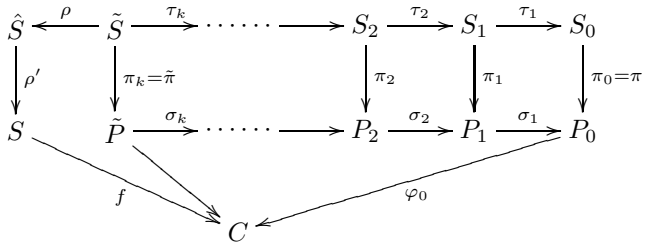
If $n_{E_{i+1}} = n_{E_i}$, then $n_{E_l} = n_{E_{i+2}} = 2n_{E_i} < 3n_{E_i}$, a contradiction. So $n_{E_i} < n_{E_{i+1}} \leq 2n_{E_i}$, that is, there is else $E_k$ pass through $p_i$ $(k < i)$. Then $n_{E_l} = 2n_{E_i} + n_{E_k} \leq 3n_{E_i}$. implies that $n_{E_l} = 3n_{E_i}$. Thus $n_{C_1} = n_\Gamma$. By the

above discussion, the equality implies that $C_1$ pass through only one node in the fiber $F_{j-1} \subseteq S_{j-1}$. Let $E_{i+3}$ be the exceptional component of $\sigma_{i+3}$ at $p_{i+2}$ and $\mathcal{E}_{i+3}$ be the totally transform. Since $\Gamma$ intersects $C_1$ in $F_{j-1}$, all components of $\tilde{\pi}^*\mathcal{E}_{i+3}$ must be contracted by $\rho$ (if not, we can find a component $C_2$ of $\tilde{\pi}^*\mathcal{E}_{i+3}$ such that $C_2$ is not contracted $\rho$ and $C_2$ cross the $C_1$ in $F_{j-1}$, $C_1$ has two nodes, a contradiction.) Then after contracting all components of $\tilde{\pi}^*\mathcal{E}_{i+3}$ and $\tilde{\pi}^*E_{i+1}$, $C_1$ becomes $(-1)$-curve. It implies that $p_i$ is not a singular point of branch locus, a contradiction.

Thus we prove that $\Gamma$ is a $(-1)$-curve. □

# 6 Classification of singular points of branch locus

In Sect. 2.4, we have give a simple numerical classification of singular points of branch locus according to $(Z_1^{(i)})^2$. Let us recall the results.




where $\tilde{\pi}$ is the canonical resolution of $\pi$. Let $E_i$ be the exceptional curve of $\sigma_i$, and $\mathcal{E}_{i+1}$ be the totally transform of $E_i$ in $\tilde{P}$. Let $\sigma = \sigma_1 \cdots \sigma_k : \tilde{P} \to P_0$.

$\rho : \tilde{S} \to \hat{S}$ contracts all $(-1)$-curves in the exceptional curves of the singularities of $\pi$ in the canonical resolution.

Any singular point $p_i$ has a decomposition $\tilde{\pi}^*\mathcal{E}_{i+1} = Z_1^{(i)} + Z_2^{(i)} + Z_3^{(i)}$.

**Case** 1. $(Z_1^{(i)})^2 = (Z_2^{(i)})^2 = (Z_3^{(i)})^2 = -1$.

As $2Z_2^{(i)} \leq \tilde{\pi}^*\mathcal{E}_{i+1} \leq \tilde{F}$, $\rho$ contracts all components of $\mathrm{supp}(Z_2^{(i)})$. Hence $p_a(Z_2^{(i)}) = p_a(Z_3^{(i)}) = 0$.

(1.1) $p_i$ is totally ramified and $\tilde{\pi}^*E_{i+1} = 3C_1$.

In this subcase, $Z_1^{(i)} \geq Z_2^{(i)} \geq Z_3^{(i)}$ and $Z_3^{(i)}C_1 = -1$. In particular, if $p_a(Z_1^{(i)}) = 0$, $p_i$ is just a good cusp.

By a discussion in Sect. 5.3, $Z_3^{(i)} = C_1$. If $Z_1^{(i)} - C_1$ is not connect, then $Z_1^{(i)} - C_1 = A + B$ where $A, B \geq 0$ and $A, B$ have no common component. As $(Z_1^{(i)} - C_1)^2 = -2$, $A^2 = B^2 = -1$. Since $(A + C_1)^2 < 0$ and $(A - C_1)^2 < 0$, $C_1A = 0$. Similarly, we have $Z_1^{(i)}A = 0$. Then $A^2 = Z_1^{(i)}A - C_1A = 0$, a contradiction. So $Z_1^{(i)} - C_1$ is connect.

Let $p_{i+1}$ be a singular point in $E_{i+1}$. As $Z_1^{(i)} - C_1$ is connect and doesn't contain $C_1$, $\mathrm{supp}(Z_1^{(i)} - C_1) = \mathrm{supp}(Z_1^{(i+1)})$, that is, $p_{i+1}$ is unique one singular point of branch locus in $E_{i+1}$. Blow up $p_{i+1}$, we get the exceptional component $E_{i+2}$. By the above discussion in Sect. 5.3, $E_{i+2}$ is totally ramified component. Moreover, there are two singular points in $E_{i+1}$, one of them is $p_{i+2} = E_{i+1} \cap$

$E_{i+2}$. After blowing up the two singular points, both $E_{i+1}$ and $E_{i+2}$ become isolated $(-3)$-curves.

(1.2) $p_i$ is simply ramified and $\tilde{\pi}^* E_{i+1} = 2C_1 + C_2$.

In this case $Z_1^{(i)} \geq Z_2^{(i)}$. $Z_2^{(i)} C_1 = -1$ and $Z_3^{(i)} C_2 = -1$. Similarly, there is unique one singular point $p_{i+1}$ of branch locus in $E_{i+1}$. After blowing up $p_{i+1}$, $E_{i+1}$ becomes an isolated $(-2)$-curves.

**Case** 2. $(Z_1^{(i)})^2 = -2$.

(2.1) $p_i$ is totally ramified and $E_{i+1}$ is a simply ramified exceptional component.

In this case, $(Z_2^{(i)})^2 = -1$ and $Z_3^{(i)} = 0$. Let $\tilde{\pi}^* E_{i+1} = 2C_1 + C_2$. By the above discussion in 5.3, $Z_2^{(i)} = C_1$, and there is unique one singular point $p_{i+1}$ of branch locus in $E_{i+1}$. After blowing up $p_{i+1}$, $E_{i+1}$ becomes an isolated $(-2)$-curves.

(2.2) $p_i$ is totally ramified and $E_{i+1}$ is a totally ramified exceptional component. However, this kind of singularity does not exist in the case that $f$ is a semistable fibration.

Suppose that these exists the type of singularity. One can assert that there are only two singular point $p_{i+1}$ of branch locus in $E_{i+1}$.After blowing up the two singular points, $E_{i+1}$ becomes an isolated $(-3)$-curves.

Let $\tilde{F}_0$ be the image of fiber $F_0$ in some ruled surface $P_j$ and let $p_0 \in \tilde{F}_0$ be the above singularity. Then by a resolution and contracting $(-1)$-curve, one can find that the image fiber $F$ of fibration $f : S \to C$ has a singularity which is not a node, a contradiction.

(2.3) $p_i$ is simplify ramified.

In this case $(Z_3^{(i)})^2 = -1$ and $Z_2^{(i)} = 0$, and of $Z_3^{(i)}$, $Z_1^{(i)}$ and $Z_3^{(i)}$ have no common component. We assert that $E_{i+1}$ doesn't lie in branch locus. In fact, if $E_{i+1}$ lies in branch locus, then $E_{i+1}$ is simply ramified, and $\tilde{\pi}^* E_{i+1} = 2C_1 + C_2$. By the discussion in sect. 5.3, $C_1$ is a $(-1)$-curve. If $Z_1^{(i)} C_1 < 0$, then $(Z_1^{(i)} - C_1) C_k \leq 0$ for all irreducible components $C_k$ of $Z_1^{(i)}$. By the definition of fundamental cycle, we have $Z_1^{(i)} - C_1 \geq Z_1^{(i)}$, a contradiction. So $Z_1^{(i)} C_1 = 0$. Thus $\tilde{\pi}^* \mathcal{E}_{i+1} C_1 = Z_1^{(i)} C_1 = 0$. However, $\tilde{\pi}^* \mathcal{E}_{i+1} C_1 = \mathcal{E}_{i+1} \tilde{\pi}_* C_1 = -1$, a contradiction.

Therefore, $E_{i+1}$ doesn't lie in branch locus.

**Case** 3. $(Z_1^{(i)})^2 = -3$.

In this case, we have $Z_2^{(i)} = Z_3^{(i)} = 0$, and $p_i$ must be totally ramified.

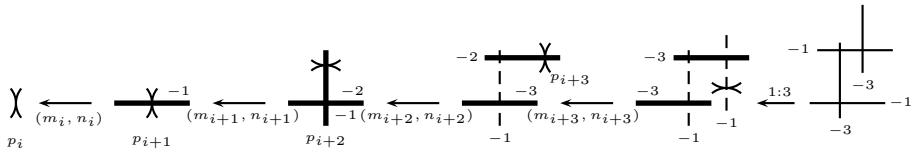
By a similarly discussion as case (2.3), we can assert that $E_{i+1}$ doesn't lie in branch locus.

Up to now we can dived all singular points of branch locus into 7 classes as follows.

To illustrate the canonical resolution of singularities, thick line is used to represent the component contained in the triple ramification divisor $D_2$. Thin line is used to represent the component contained in the double ramification divisor $D_1$, and dashed line is used to represent the component not contained in the branch locus. The self-intersection number is marked near the component. The pair of numbers under an arrow $\underset{(m_i, n_i)}{\longleftarrow}$ represents the invariants $m_i, n_i$ associated with this blowing-up. And $\overset{1:3}{\longleftarrow}$ represents a triple cover.

Each singular point $p$ of branch locus (including its infinitely close singular points occurring in the canonical resolution) is one of the following types:

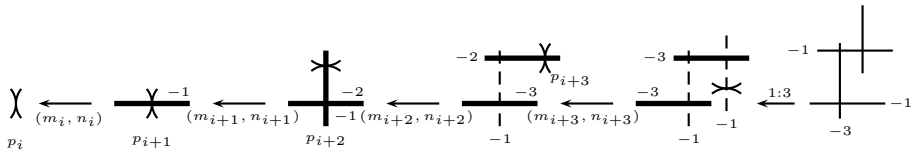
**Type** 0 **:** $p_i$ is a good cusp, that is, $\tilde{\pi}^*\mathcal{E}_{i+1} = Z_1^{(i)}+Z_2^{(i)}+Z_3^{(i)}$, $(Z_j^{(i)})^2 = -1$ and $p_a(Z_j^{(i)}) = 0$ $(j = 1, 2, 3)$, and $p_i$ is totally ramified.




where $m_i = m_{i+1} = m_{i+3} = n_{i+3} = 1$, $n_i = -1$, $n_{i+1} = m_{i+2} = 0$, $n_{i+2} = 2$.

In what follows, $p_i$ as well as its infinitely close singular points $p_{i+1}$, $p_{i+2}$, $p_{i+3}$ is regarded as one singular point, also denoted by $p_i$. Then $p_i$ contributes 4 contractible curves in the exceptional set of the above resolution.

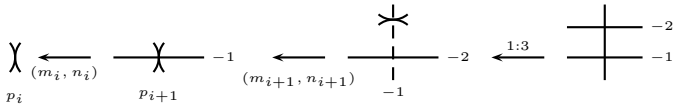
**Type** 1 $(m, n)$**:** $p_i$ is totally ramified and $(Z_1^{(i)})^2 = -1$ such that $(Z_j^{(i)})^2 = -1$ and $p_a(Z_j^{(i)}) = 0$ $(j = 2, 3)$. However, $p_a(Z_1^{(i)}) \neq 0$.




where $m_i = m_{i+1} = m_{i+3} = m$, $m_{i+2} = 0$; $n_i = n$, $n_{i+1} = n + 1$, $n_{i+2} = 2$ and $n_{i+3} = n + 2$.

$p_i$ as well as its infinitely close singular points $p_{i+1}$, $p_{i+2}$, $p_{i+3}$ is regarded as one singular point, also denoted by $p_i$. Then $p_i$ contributes 3 contractible curves in the exceptional set of the above resolution.

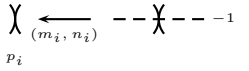
**Type** 3 $(m, n)$**:** $p_i$ is totally ramified and $(Z_1^{(i)})^2 = -2$ such that $(Z_2^{(i)})^2 = -1$ and $p_a(Z_2^{(i)}) = 0$. $E_{i+1}$ is simply ramified.




where $m_i = m$, $m_{i+1} = m + 1$; $n_i = n_{i+1} = n$.

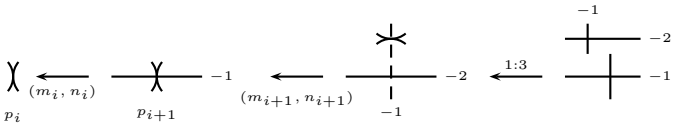
$p_i$ as well as its infinitely close singular points $p_{i+1}$ is regarded as one singular point, also denoted by $p_i$. Then $p_i$ contributes only one contractible curves in the exceptional set of the above resolution.

**Type** 4 $(m, n)$**:** $p$ is totally ramified and $(Z_1^{(i)})^2 = -3$.




where $m_i = m$, $n_i = n$. $p_i$ contributes no contractible curve in the exceptional set of the above resolution.

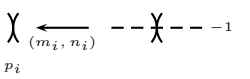
**Type** 5 $(m,0)$**:** $p$ is simply ramified and $(Z_1^{(i)})^2=-1$ such that $(Z_j^{(i)})^2=-1$ and $p_a(Z_j^{(i)})=0$ $(j=2,3)$. However, $p_a(Z_1^{(i)})\neq 0$. $Z_1^{(i)}$ and $Z_3^{(i)}$ have no common component.




where $m_i=m$, $m_{i+1}=m+1$; $n_i=n_{i+1}=0$. .

$p_i$ as well as its infinitely close singular points $p_{i+1}$ is regarded as one singular point, also denoted by $p_i$. Then $p_i$ contributes 3 contractible curves in the exceptional set of the above resolution

**Type** 6 $(m,0)$**:** $p$ is simply ramified and $(Z_1^{(i)})^2=-2$ such that $Z_2^{(i)}=0$, and $(Z_3^{(i)})^2=-1$, $p_a(Z_3^{(i)})=0$ . $Z_1^{(i)}$ and $Z_3^{(i)}$ have no common component.




where $m_i=m$, $n_i=0$. $p_i$ contributes only one contractible curve in the exceptional set of the above resolution.

In fact, the singular point of type 5 $(m,0)$ (resp. type 6 $(m,0)$) is just $(2m+1\to 2m+1)$ singularity (resp. singularity of order $2m$). The details can be referred to [23].

The number of singular points of type $k$ $(m,n)$ (resp. type 2 $(m,n,m',n')$) occurring the canonical resolution is denoted by $s_k(m,n)$. $s_0$ is the number of good cusps.

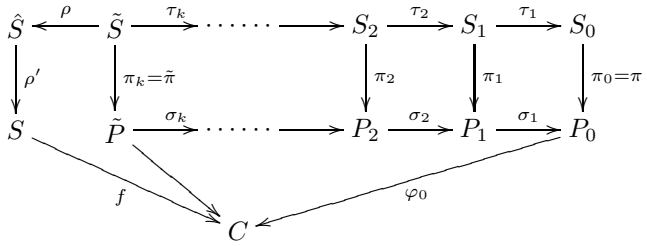
These $s_k(m,n)$ are called singularity counts.

Recall that $\varepsilon_P$ is the number of the curves contracted by $\rho$. By the above discussion, we have

$$\begin{aligned}\varepsilon_P &= 4s_0+3\sum_{m,n}s_1(m,n)+\sum_{m,n}s_3(m,n)\\ &\quad +3\sum_{m}s_5(m,0)+\sum_{m}s_6(m,0).\end{aligned}$$

# 7 Global invariants of a trigonal fibration

Recall the commutative diagram (11).




where $\tilde{\pi}$ is the canonical resolution of $\pi$. Let $E_i$ be the exceptional curve of $\sigma_i$, and $\mathcal{E}_{i+1}$ be the totally transform of $E_i$ in $\tilde{P}$. Let $\sigma=\sigma_1\cdots\sigma_k:\tilde{P}\to P_0$. $\tilde{R}=\tilde{D}_1+2\tilde{D}_2$ is the branch locus of $\tilde{\pi}$.

$\rho : \tilde{S} \to \hat{S}$ contracts all $(-1)$-curves in the exceptional curves of the singularities of $\pi$ in the canonical resolution. $\rho'$ contracts all $(-1)$-curves in fibers of $f\rho' : \hat{S} \to C$. $\rho$ and $\rho'$ consist of some blown-downs, that is, $\rho = \rho_r \rho_{r-1} \cdots \rho_1$ and $\rho' = \rho_{r+r'} \cdots \rho_{r+1}$.

$$\begin{array}{ccccccccccccc}
\tilde{S} & \xrightarrow{\rho_1} & S'_1 & \xrightarrow{\rho_2} & \cdots\cdots & S'_r(=\hat{S}) & \xrightarrow{\rho_{r+1}} & S'_{r+1} & \xrightarrow{\rho_{r+2}} & \cdots\cdots & S'_{r+r'-1} & \xrightarrow{\rho_{r+r'}} & S \\
\downarrow{\scriptstyle \tilde{f}=f_0} & & \downarrow{\scriptstyle f_1} & & & \downarrow{\scriptstyle f_r} & & \downarrow{\scriptstyle f_{r+1}} & & & \downarrow{\scriptstyle f_{r+r'-1}} & & \downarrow{\scriptstyle f} \\
C & = & C & & \cdots\cdots & C & = & C & & \cdots\cdots & C & = & C
\end{array}$$

Put $\varepsilon_P = r$, $\varepsilon_F = r'$, $\varepsilon = r + r'$.

By Corollary 4.1, formulae (17) (18) and recall that $r_{\tilde{D}_i} = -2t_i$, we have

$$\begin{aligned}
3(g+1)K_f^2 &= \sum_{i=0}^{k} c(m_i, n_i) + 3(g+1)\varepsilon \\
&\quad - \left(3g - 1 - \frac{g-3}{2g+3-2\alpha}\right) t_1 - 8(g-1)t_2 \qquad (22) \\
e_f &= -2t_1 - 4t_2 + 3k - \varepsilon, \qquad (23)
\end{aligned}$$

where $t_i$ is the number of all exceptional components of the blowing-ups $\sigma : \tilde{P} \to P_0$ which lie in $\tilde{D}_i$, and where

$$c(m,n) = \frac{(2g-6)(g+2-\alpha-m)m}{2g+3-2\alpha} - 8(m+n)^2 + 8(g+2)(m+n) - 9(g+1).$$

By the classification of singular points of branch locus, we have

$$\begin{aligned}
t_1 &= \sum_{m,n} s_3(m,n) + \sum_{m} s_5(m,0) \\
t_2 &= 2s_0 + 2\sum_{m,n} s_1(m,n) \\
\varepsilon_P &= 4s_0 + 3\sum_{m,n} s_1(m,n) + \sum_{m,n} s_3(m,n) \\
&\quad + 3\sum_{m} s_5(m,0) + \sum_{m} s_6(m,0)
\end{aligned}$$

where $\sum\limits_{m,n}$ runs over all possible values of local invariants for $m_i, n_i$ (see Sect. 2.3).

The sum $\sum\limits_{i=0}^{k} c(m_i, n_i)$ also can be computed locally by each type of singular point. For example, for a singular point $p_i$ of type 1 $(m,n)$ which contains its infinitely near singular point $p_{i+1}$ with resolution invariants $(m_{i+1}, n_{i+1}) = (m, n+1)$, $p_{i+2}$ with $(m_{i+2}, n_{i+2}) = (0,2)$, and $p_{i+3}$ with $(m_{i+3}, n_{i+3}) = (m, n+2)$ (see Sect. 6). Its contribution to $\sum\limits_{i=0}^{k} c(m_i, n_i)$ is just $c(m,n) + c(m,n+1) + c(0,2) + c(m,n+2) = 3c(m,n+1) + 7g - 25$.

Note that $\varepsilon = \varepsilon_P + \varepsilon_F$ and

$$\begin{aligned} k = &4s_0 + 4\sum_{m,n} s_1(m,n) + 2\sum_{m,n} s_3(m,n) \\ &+ \sum_{m,n} s_4(m,n) + 2\sum_{m} s_5(m,0) + \sum_{m} s_6(m,0), \end{aligned}$$

we have

$$\begin{aligned} e_f =&\sum_{m,n} s_1(m,n) + 3\sum_{m,n} s_3(m,n) \\ &+ 3\sum_{m,n} s_4(m,n) + \sum_{m} s_5(m,0) + 2\sum_{m} s_6(m,0) - \varepsilon_F \qquad (24) \\ 3(g+1)K_f^2 =&\frac{6(g-3)(g+1-2\alpha)}{2g+3-\alpha} s_0 + \sum_{m,n} c_1(m,n)s_1(m,n) \\ &+ \sum_{m,n} c_3(m,n)s_3(m,n) + \sum_{m,n} c_4(m,n)s_4(m,n) \\ &+ \sum_{m} c_5(m,0)s_5(m,0) + 2\sum_{m} c_6(m,0)s_6(m,0) + 3(g+1)\varepsilon_F, \qquad (25) \end{aligned}$$

where

$$\begin{aligned} c_1(m,n) =&3c(m,n+1), \\ c_3(m,n) =&2c\left(m+\frac{1}{2},n\right), \\ c_4(m,n) =&c(m,n), \\ c_5(m,0) =&6g+6+2c\left(m+\frac{1}{2},0\right), \\ c_6(m,0) =&3g+3+c(m,0). \end{aligned}$$

It is easy to verify that the coefficients $c_k(m,n)$ of each type of singular point are non-negative integers. By Lemma 5.1, $\varepsilon_F = 0$ whenever $\alpha \geq 2$.

# 8 The proof of Theorem 1.1

In what follows we assume that $\alpha \geq 2$ and $g \geq 4$. Let $Q(x,y) = \frac{\alpha(\alpha-1)}{2}(x+y)^2 - (\alpha-1)(g-3)(x+y)y + \frac{g(g-3)}{2}y^2$ where $x, y \in \mathbb{Z}$. One can see easily that it is a positive definite quadratic form. Let $q = \min\limits_{y \equiv \alpha \pmod 2} Q(x,y)$.

Put

$$\lambda_0 = 12 - \frac{24(\alpha-1)\alpha}{\alpha g(g+3\alpha-6) - 2q}$$

In order to prove $\lambda_f \leq \lambda_0$, we only need to prove $\frac{\lambda_0}{12-\lambda_0} e_f \geq K_f^2$, by Noether's formula. Set $\mu_0 = \frac{\lambda_0}{12-\lambda_0}$.

Recall formula (16), that is,

$$\frac{\alpha}{4g+6-4\alpha}r_{D_1}+\frac{g+2-\alpha}{\alpha-1}r_{D_2}-D_1D_2=0.$$

Let

$$u(m,n)=\frac{\alpha}{4g+6-4\alpha}(4m^2-2m)+\frac{g+2-\alpha}{\alpha-1}(n^2-n)-2mn.$$

By formula (17), (18) and (5), we have

$$\begin{aligned}0=&3u(1,0)s_0+\sum_{m,n}u_1(m,n)s_1(m,n)\\&+\sum_{m,n}u_3(m,n)s_3(m,n)+\sum_{m,n}u_4(m,n)s_4(m,n)\\&+\sum_{m}u_5(m,0)s_5(m,0)+2\sum_{m}u_6(m,0)s_6(m,0),\end{aligned}$$

where

$$\begin{aligned}u_1(m,n)=&3u(m,n+1),\\u_3(m,n)=&2u\left(m+\frac{1}{2},n\right),\\u_4(m,n)=&u(m,n),\\u_5(m,0)=&2u\left(m+\frac{1}{2},0\right),\\u_6(m,0)=&u(m,0).\end{aligned}$$

Set $\theta_0=\frac{2(g-3)(g+1-\alpha)}{3\alpha(g+1)}$. In order to prove $\mu_0 e_f\geq K_f^2$, we only need to show that

$$0\leq\mu_0e_f-K_f^2+\theta_0\left(3u(1,0)s_0+\sum_k\sum_{m,n}u_k(m,n)s_k(m,n)\right),$$

that is,

$$\begin{aligned}0\leq&\sum_{m,n}\left(\mu_0-\frac{c_1(m,n)}{3g+3}+\theta_0u_1(m,n)\right)s_1(m,n)\\&+\sum_{m,n}\left(3\mu_0-\frac{c_3(m,n)}{3g+3}+\theta_0u_3(m,n)\right)s_3(m,n)\\&+\sum_{m,n}\left(3\mu_0-\frac{c_4(m,n)}{3g+3}+\theta_0u_4(m,n)\right)s_4(m,n)\\&+\sum_{m}\left(\mu_0-\frac{c_5(m,0)}{3g+3}+\theta_0u_5(m,0)\right)s_5(m,0)\\&+\sum_{m}\left(2\mu_0-\frac{c_6(m,0)}{3g+3}+\theta_0u_6(m,0)\right)s_6(m,0),\end{aligned}$$

by invariants formula (24).

For simplicity, we set $\bar{m} = g + 2 - \alpha - 2m$ and $\bar{n} = \alpha - 2n$. From a direct computation, we have

$$-\frac{c(m,n)}{3g+3} + \theta_0 u(m,n) = -\frac{\eta_0}{3} + \frac{Q(\bar{m},\bar{n})}{3\alpha(\alpha-1)},$$

where $\eta_0 = \frac{3g-2}{2} + \frac{g(g-3)}{2\alpha-2}$.

Thus the above inequality is writen as

$$\begin{aligned}
0 \leq & \sum_{m,n} \left( \mu_0 - \eta_0 + \frac{Q(\bar{m},\bar{n}-2)}{\alpha(\alpha-1)} \right) s_1(m,n) \\
& + \sum_{m,n} \left( 3\mu_0 - \frac{2\eta_0}{3} + \frac{2Q(\bar{m}-1,\bar{n})}{3\alpha(\alpha-1)} \right) s_3(m,n) \\
& + \sum_{m,n} \left( 3\mu_0 - \frac{\eta_0}{3} + \frac{Q(\bar{m},\bar{n})}{3\alpha(\alpha-1)} \right) s_4(m,n) \\
& + \sum_{m} \left( \mu_0 - \frac{2\eta_0}{3} - 2 + \frac{2Q(\bar{m}-1,\alpha)}{3\alpha(\alpha-1)} \right) s_5(m,0) \\
& + \sum_{m} \left( 2\mu_0 - \frac{\eta_0}{3} - 1 + \frac{Q(\bar{m},\alpha)}{3\alpha(\alpha-1)} \right) s_6(m,0),
\end{aligned}$$

In order to prove the inequality, it is enough to verify that coefficient of each singularity count $s_k$ is non-negative. Namely, we need verify the following inequality.

(1) $\mu_0 - \eta_0 + \frac{Q(\bar{m},\bar{n}-2)}{\alpha(\alpha-1)} \geq 0$

(2) $3\mu_0 - \frac{2\eta_0}{3} + \frac{2Q(\bar{m}-1,\bar{n})}{3\alpha(\alpha-1)} \geq 0$

(3) $3\mu_0 - \frac{\eta_0}{3} + \frac{Q(\bar{m},\bar{n})}{3\alpha(\alpha-1)} \geq 0$

(4) $\mu_0 - \frac{2\eta_0}{3} - 2 + \frac{2Q(\bar{m}-1,\alpha)}{3\alpha(\alpha-1)} \geq 0$

(5) $2\mu_0 - \frac{\eta_0}{3} - 1 + \frac{Q(\bar{m},\alpha)}{3\alpha(\alpha-1)} \geq 0.$

As $Q(x,y) \geq q$ and $\frac{Q(x,\alpha)}{\alpha(\alpha-1)} = \frac{1}{2}(x + \alpha - g + 3)^2 + \frac{3(g-3)}{2} + \frac{g(g-3)}{2\alpha-2}$, we only need to show the following inequalities holds.

(1′) $\mu_0 - \eta_0 + \frac{q}{\alpha(\alpha-1)} \geq 0$

(2′) $3\mu_0 - \frac{2\eta_0}{3} + \frac{2q}{3\alpha(\alpha-1)} \geq 0$

(3′) $3\mu_0 - \frac{\eta_0}{3} + \frac{q}{3\alpha(\alpha-1)} \geq 0$

(4′) $\mu_0 - \frac{2\eta_0}{3} - 2 + \frac{2}{3}(\frac{3(g-3)}{2} + \frac{g(g-3)}{2\alpha-2}) \geq 0$

(5′) $2\mu_0 - \frac{\eta_0}{3} - 1 + \frac{1}{3}(\frac{3(g-3)}{2} + \frac{g(g-3)}{2\alpha-2}) \geq 0.$

It is obvious that (1′) implies (2′) as well as (3′), and, (4′) implies (5′). From a straightforward computation, we know $\mu_0 - \eta_0 + \frac{q}{\alpha(\alpha-1)} = 0$. So (1′) holds.

We verify inequality (4′). As $\mu_0 = \eta_0 - \frac{q}{\alpha(\alpha-1)}$ and $q \leq Q(-1,1) = \frac{g(g-3)}{2}$,

we have

$$\begin{aligned}
&\mu_0 - \frac{2\eta_0}{3} - 2 + \frac{2}{3}\left(\frac{3(g-3)}{2} + \frac{g(g-3)}{2\alpha-2}\right) \\
&= \frac{\eta_0}{3} - \frac{q}{\alpha(\alpha-1)} - 2 + \frac{2}{3}\left(\frac{3(g-3)}{2} + \frac{g(g-3)}{2\alpha-2}\right) \\
&\geq \frac{\eta_0}{3} - \frac{g(g-3)}{2\alpha(\alpha-1)} - 2 + \frac{2}{3}\left(\frac{3(g-3)}{2} + \frac{g(g-3)}{2\alpha-2}\right) \\
&= \frac{9g-32}{6} + \frac{g(g-3)}{2\alpha} \geq 0.
\end{aligned}$$

Up to now we complete the proof of Theorem 1.1.

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